EE 317
Spring 2001
Midterm Exam #2

Name: KEY

ID Number: ____________

This exam consists of 6 pages.
The problems are worth a total of 100 points. The number of points you can earn on each problem is given in [brackets] next to the problem number.
For all problems, clearly show **all** work.
Where appropriate, enclose final answers in boxes, and include units in all answers.
Calculators are permitted.
You are permitted to consult your textbook but not any other notes of any kind.

1. [2 pts/characteristic] A certain signal $x(t)$ has the Fourier series in trigonometric form given by:

$$x(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4\pi n} \sin(40\pi n t)$$

Circle the appropriate characteristics of the signal based upon what you know about its Fourier series:

conflict exam: even

even function	odd function	- sine function is odd; sum of sines is odd
periodic	aperiodic	- any signal that has a Fourier series representation is periodic
energy signal	power signal	- bounded periodic signals are power signals (∞ energy, finite power)
continuous-frequency spectrum	discrete-frequency spectrum	- frequency components only at multiples of ω_0 ($\omega_0 = 40\pi$ in this case)
real-valued	complex-valued	- sine is real-valued; Fourier coef's (c_n) in this case are real-valued (no j's anywhere!)

2. [0, 4 points] What is the fundamental period of the signal in Prob. 1?

a) $1/4\pi$ sec b) $1/40\pi$ sec c) 25 msec

d) 50 msec e) 40π sec f) not enough information to solve

$\omega_0 = 40\pi \quad T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{40\pi} = \frac{1}{20} = 0.05$ sec

$= 50$ msec

conflict exam: $T_0 = 25$ ms

For Problems 3-6, refer to the following exponential Fourier series of the signal $f(t)$. Note that the summation has a finite number of terms.

$$f(t) = 0.3 + \sum_{\substack{n=-2 \\ n\neq 0}}^{2} \frac{j(-1)^n}{n|n|} e^{j6280nt}$$

$\omega_o = 6280 \text{ rad/s}$

$f_o = \frac{\omega_o}{2\pi} = \frac{6280}{2\pi} = 999 \text{ Hz}$

3. [0, 2, 4 points] What is the total normalized average power of the signal?

$P = c_o^2 + 2|c_1|^2 + 2|c_2|^2$

$= 0.3^2 + 2\left(\frac{1}{1^2}\right)^2 + 2\left(\frac{1}{2^2}\right)^2$

$\boxed{P = 2.215 \text{ W}}$

$c_o = 0.3$

$|c_n| = \left|\frac{j(-1)^n}{n|n|}\right| = \frac{1}{n^2}$

conflict exam: $P = 2.285$

4. [0, 4 points] What percentage of the total power in the signal lies below 1200 Hz?

a) 0 % b) 4.1 % c) 60 % d) 90 % e) 94 % f) 100 %

$\%\text{ power} = (100\%)\frac{0.3^2 + 2(1)^2}{2.215} = 94\%$

conflict exam: 94.5%

signal has frequency components at $f = (0)(999), (1)(999),$ and $(2)(999)$ $= 0, 999, 1998 \text{ Hz}$

5. [0, 4 points] What is the time-average value of the signal?

a) 0 b) 0.09 c) 0.15 d) 0.3 e) 0.188 f) not enough info

$f(t)|_{avg} = C_o = 0.3$

conflict exam: 0.4

6. [2 pts each] What is the phase (in radians) of each coefficient in the Fourier series?

$\angle C_{-2} =$ $-\pi/2$

$\angle C_{-1} =$ $\pi/2$

$\angle C_0 =$ 0

$\angle C_1 =$ $-\pi/2$

$\angle C_2 =$ $\pi/2$

$C_{-n} = \frac{j(-1)^{-2}}{-2|-2|} = \frac{j(1)}{-2(2)} = -j\frac{1}{4} = \frac{1}{4}\angle -\pi/2$

$C_{-1} = j(-1)^{-1}/-1|-1| = -j/-1 = j = 1\angle \pi/2$

$C_o = 0.3$ (by inspection) $= 0.3\angle 0$ (real)

$C_1 = j(-1)^1/1|1| = -j = 1\angle -\pi/2$

$C_2 = \frac{j(-1)^2}{2|2|} = \frac{j}{4} = \frac{1}{4}\angle \pi/2$

conflict exam: same answers

7. [0-16 points] Find the Fourier transform of the following signal. You may use the direct integration approach, or you may use Tables 5.1 and 5.2 in your text to minimize the work required. (For full credit, you must show **all** of your work.)

$$f(t) = e^{-at}\cos(\omega_o t)\,u(t)$$

Let $f_1(t) = e^{-at}u(t) \Leftrightarrow F_1(\omega) = \frac{1}{a+j\omega}$ ← Table 5.2

Let $f_2(t) = \cos(\omega_o t) \Leftrightarrow F_2(\omega) = \pi[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]$

$f(t) = f_1(t) f_2(t) \Leftrightarrow F(\omega) = \frac{F_1(\omega) * F_2(\omega)}{2\pi}$ (convolution prop.) Table 5.1

$$\Rightarrow F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(x)\,F_2(\omega - x)\,dx \qquad (x = \text{dummy variable})$$

$$= \frac{1}{2\cancel{\pi}}\int_{-\infty}^{\infty}\frac{1}{a+jx}\cancel{\pi}[\delta(\omega-x-\omega_o)+\delta(\omega-x+\omega_o)]\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\delta[(\omega-\omega_o)-x]}{a+jx}dx + \frac{1}{2}\int_{-\infty}^{\infty}\frac{\delta[(\omega+\omega_o)-x]}{a+jx}dx$$

$$= \frac{1}{2}\frac{1}{a+j(\omega-\omega_o)} + \frac{1}{2}\frac{1}{a+j(\omega+\omega_o)}$$ ← far enough for full credit.

$$= \frac{1}{2}\frac{a+j(\omega+\omega_o)+a+j(\omega-\omega_o)}{[a+j(\omega-\omega_o)][a+j(\omega+\omega_o)]}$$

$$= \frac{1}{\cancel{2}}\frac{\cancel{2}a+j\cancel{2}\omega}{(a+j\omega-j\omega_o)(a+j\omega+j\omega_o)} = \frac{a+j\omega}{(a+j\omega)^2+(a+j\omega)(\cancel{j\omega_o-j\omega_o})+\omega_o^2}$$

0

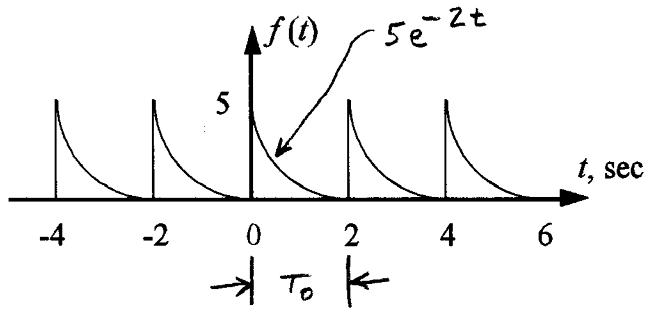
$$\boxed{F(\omega) = \frac{a+j\omega}{(a+j\omega)^2+\omega_o^2}}$$

← transform no. 7 on page facing inside front cover of text ("e^{at}" should be "e^{-at}")

conflict exam:

$$F(\omega) = \frac{2+j\omega}{(2+j\omega)^2+(5\pi)^2}$$

8. [0-16 points] Find the Fourier series coefficients for the following periodic signal,



$T_o = 2$ sec.

$\omega_o = \frac{2\pi}{T_o} = \pi$

which can be expressed mathematically in the time domain as

$$f(t) = \sum_{n=-\infty}^{\infty} 5e^{-2(t-2n)} \operatorname{rect}\left(\frac{t-1-2n}{2}\right)$$

eqn (4.23): $c_k = \frac{1}{T_o} \int_{\langle T_o \rangle} f(t)\, e^{-jk\omega_o t}\, dt$

$$c_k = \frac{1}{2} \int_0^2 5e^{-2t} e^{-jk\pi t}\, dt$$

$$= \frac{5}{2} \int_0^2 e^{-(2+jk\pi)t}\, dt$$

$$= \frac{5}{2} \frac{-1}{2+jk\pi} e^{-(2+jk\pi)t} \Big|_0^2$$

$$= -\frac{5}{2} \frac{1}{2+jk\pi} \left[e^{-(2+jk\pi)(2)} - 1 \right]$$

$$= \frac{-5}{4+jk2\pi} \left(e^{-4} e^{-jk2\pi} - 1 \right)$$

Since $e^{-jk2\pi} = 1$, then

$$\boxed{c_k = \frac{5}{4+jk2\pi}\left(1 - e^{-4}\right)}$$

$$c_o = \frac{1}{T_o} \int_{\langle T_o \rangle} f(t)\, dt$$

$$= \frac{1}{2} \int_0^2 5e^{-2t}\, dt$$

$$= \frac{5}{2}\left(\frac{-1}{2}\right) e^{-2t} \Big|_0^2$$

$$= -\frac{5}{4}\left(e^{-2(2)} - 1\right)$$

$$\boxed{c_o = \frac{5}{4}\left(1 - e^{-4}\right)}$$

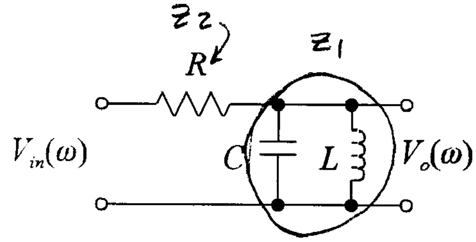
$$= 1.227$$

conflict exam:

$$c_k = \frac{6}{4+jk\pi}\left(1 - e^{-8}\right)$$

4

$$c_o = \frac{6}{4}\left(1 - e^{-8}\right) = 1.50$$



For Problems 9-12, refer to the *RLC* circuit to the left.

$$Z_1 = \frac{1}{j\omega C} \,\|\, j\omega L = \frac{\left(\frac{1}{j\omega C}\right)(j\omega L)}{\frac{1}{j\omega C} + j\omega L} = \frac{L/C}{j\left(\frac{-1}{\omega C} + \omega L\right)}$$

$$Z_2 = R$$

9. [12 points] Derive the transfer function $H(\omega)$ for the circuit.

voltage-divider rule:

$$V_o = \frac{Z_1}{Z_1 + Z_2} V_{in} \Rightarrow H(\omega) = \frac{V_o}{V_{in}} = \frac{\frac{L/C}{j(1/\omega C + \omega L)}}{R + \frac{L/C}{j\left(\frac{-1}{\omega C} + \omega L\right)}} = \frac{Z_1}{Z_2 + Z_1}$$

Multiply numerator and denom. by $1/Z_1$:

$$H(\omega) = \frac{1}{1 + Z_2/Z_1} = \frac{1}{1 + \frac{j(-1/\omega C + \omega L)R}{L/C}} = \frac{1}{1 + j\left(\frac{-R}{\omega L} + \omega RC\right)}$$

$$\boxed{H(\omega) = \frac{1}{1 + j\left(\omega RC - \frac{R}{\omega L}\right)}}$$

10. [0, 4, 8 points] At what frequency does the peak power transfer occur? The power transfer characteristic is given by $|H(\omega)|^2$.

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = \frac{1}{1 + j\left(\omega RC - \frac{R}{\omega L}\right)} \cdot \frac{1}{1 - j\left(\omega RC - \frac{R}{\omega L}\right)}$$

$$= \frac{1}{1 + \underbrace{\left(\omega RC - \frac{R}{\omega L}\right)^2}_{\text{this quantity is alway positive (because of power of 2)}}}$$

$|H|^2$ has peak value at frequency that satisfies:

$$\omega RC - \frac{R}{\omega L} = 0 \Rightarrow \omega \cancel{R} C = \frac{\cancel{R}}{\omega L} \Rightarrow \omega C = \frac{1}{\omega L} \Rightarrow \omega^2 = \frac{1}{LC}$$

$$\Rightarrow \boxed{\omega_p = \frac{1}{\sqrt{LC}}}$$

11. [0, 4, 8 points] What is/are the cut-off (half-power) frequency/frequencies of the circuit?

Half-power frequencies occur when $|H(\omega)|^2 = \frac{1}{2}|H_{max}|^2$

In this case $|H_{max}|^2 = |H(\omega_{pk})|^2 = 1$, so

$$|H(\omega_c)|^2 = \frac{1}{2} \qquad \text{where } \omega_c = \text{cut-off frequency('ies)}$$

$$\Rightarrow \quad |H(\omega_c)|^2 = \frac{1}{2} = \frac{1}{1 + \left(\omega_c RC - \frac{R}{\omega_c L}\right)^2}$$

$$\Rightarrow \quad \left(\omega_c RC - \frac{R}{\omega_c L}\right)^2 = 1 \quad \Rightarrow \quad \omega_c RC - \frac{R}{\omega_c L} = \pm 1 \quad \leftarrow \text{Mult. by } \omega_c L$$

$$\omega_c^2 RLC - R = \pm \omega_c L$$

$$RLC\omega_c^2 \mp \omega_c L - R = 0 \leftarrow \text{div. by } RLC$$

quadratic formula: $ax^2 + bx + c = 0$

$x = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$

$$\omega_c^2 \mp \frac{1}{RC}\omega_c - \frac{1}{LC} = 0$$

$$\omega_c = \frac{\pm 1/RC}{2} \pm \frac{1}{2}\sqrt{\left(\frac{1}{RC}\right)^2 + 4\left(\frac{1}{LC}\right)}$$

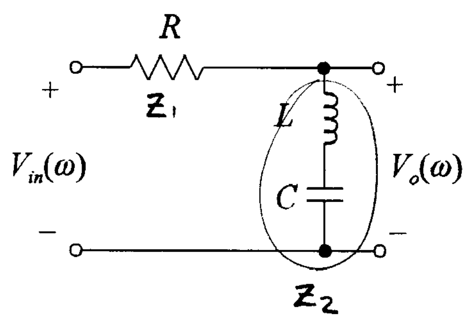
$$= \pm \frac{1}{2RC} \pm \frac{1}{2}\sqrt{\left(\frac{1}{RC}\right)^2\left[1 + \frac{4R^2C}{L}\right]}$$

$$\boxed{\omega_c = \frac{1}{2RC}\left[\pm 1 \pm \sqrt{1 + \frac{4R^2C}{L}}\right]} \quad \leftarrow \text{far enough for full credit}$$

12. [0, 4 points] What ideal filter type most closely approximates the frequency response of the circuit?

a) lowpass b) highpass c) bandpass d) bandstop

$|H(\omega)|^2$ rolls off (decreases in value) on either side of $\omega = \omega_{pk}$.



For Problems 9-12, refer to the *RLC* circuit to the left.

$$Z_2 = \frac{1}{j\omega C} + j\omega L$$

$$Z_1 = R$$

9. [12 points] Derive the transfer function $H(\omega)$ for the circuit.

$$H(\omega) = \frac{Z_2}{Z_1 + Z_2} = \frac{\frac{1}{j\omega C} + j\omega L}{R + \frac{1}{j\omega C} + j\omega L} = \boxed{\frac{1 - \omega^2 LC}{1 - \omega^2 LC + j\omega RC} = H(\omega)}$$

10. [0, 4, 8 points] At what frequency does the **minimum** power transfer occur? The power transfer characteristic is given by $|H(\omega)|^2$.

$$|H(\omega)|^2 = H(\omega)\,H^*(\omega) = \frac{1 - \omega^2 LC}{1 - \omega^2 LC + j\omega RC} \cdot \frac{1 - \omega^2 LC}{1 - \omega^2 LC - j\omega RC}$$

$$= \frac{(1 - \omega^2 LC)^2}{(1 - \omega^2 LC)^2 + (\omega RC)^2} \cdot \frac{\frac{1}{\omega^2}}{\frac{1}{\omega^2}} = \frac{\left(\frac{1}{\omega} - \omega LC\right)^2}{\left(\frac{1}{\omega} - \omega LC\right)^2 + (RC)^2}$$

$$|H(\omega)|^2 = 0 \quad \text{when} \quad \left(\frac{1}{\omega} - \omega LC\right)^2 = 0 \Rightarrow \boxed{\omega_{min} = \frac{1}{\sqrt{LC}}}$$

conflict exam

11. [0, 4, 8 points] What is/are the cut-off (half-power) frequency/frequencies of the circuit?
Hint: Peak power transfer occurs at the frequency $\omega = 0$.

Simplify $|H|^2$ further to make analysis easier:

$$|H(\omega)|^2 = \frac{\left(\frac{1}{\omega} - \omega LC\right)^2}{\left(\frac{1}{\omega} - \omega LC\right)^2 + (RC)^2} = \frac{1}{1 + \frac{(RC)^2}{\left(\frac{1}{\omega} - \omega LC\right)^2}}$$

Cut-off frequencies are those at which $|H|^2 = \frac{1}{2}|H|^2_{max}$

$$|H|^2_{max} = |H(0)|^2 = \frac{1}{1 + \underbrace{\cancel{\frac{(RC)^2}{\left(\frac{1}{0} - 0LC\right)^2}}}_{0}} = 1$$

$$\Rightarrow |H(\omega_c)|^2 = \frac{1}{2} = \frac{1}{1 + \frac{(RC)^2}{\left(\frac{1}{\omega_c} - \omega_c LC\right)^2}} \Rightarrow \frac{(RC)^2}{\left(\frac{1}{\omega_c} - \omega_c LC\right)^2} = 1$$

$$(RC)^2 = \left(\frac{1}{\omega_c} - \omega_c LC\right)^2 \rightarrow \pm RC = \frac{1}{\omega_c} - \omega_c LC \rightarrow \pm RC\omega_c = 1 - \omega_c^2 LC$$

$$LC\omega_c^2 \pm RC\omega_c - 1 = 0$$

$$\omega_c = \mp \frac{RC}{2LC} \pm \frac{1}{2LC}\sqrt{(RC)^2 + 4LC} = \mp \frac{RC}{2LC} \pm \frac{1}{2LC}\sqrt{(RC)^2\left[1 + \frac{4LC}{(RC)^2}\right]}$$

$$= \mp \frac{RC}{2LC} \pm \frac{RC}{2LC}\sqrt{1 + \frac{4L}{R^2C}} = \boxed{\frac{R}{2L}\left[\mp 1 \pm \sqrt{1 + \frac{4L}{R^2C}}\right] = \omega_c}$$

↑ 4 values: 2 are negatives of the other 2

12. [0, 4 points] What ideal filter type most closely approximates the frequency response of the circuit?

a) lowpass b) highpass c) bandpass d) bandstop

The $|H(\omega)|^2$ curve rises on either side of $\omega_{min} = \frac{1}{\sqrt{LC}}$