Homework 1: Due Feb 2nd on Dropbox in Angel

Posted on Jan 20th

Collaboration policy: You are allowed to discuss homework problems with your classmates. However, you should think about the problems before discussing them, and you must write up your own solutions. You must acknowledge any collaboration by writing your collaborators names on the front page of the assignment.

Citation policy: Try to solve the problems without reading any published literature or websites, besides the textbook and notes from class. If, however, you do use a solution or part of a solution that you found in the literature or on the web, you must cite it. Furthermore, you must write up the solution in your own words.

Submission: Homeworks typeset in latex are encouraged. If you submit a hand-written homework, please ensure that it is written legibly. The homeworks must be uploaded in the dropbox provided on Angel.

1. (a) For a binary symmetric channel with cross-over probability $p < 1/2$, for an (n, M) code $\mathcal{C}$, prove that the maximum likelihood decoder $D_{ML} : \Phi^n \to \mathcal{C}$ corresponds to the operation

$$D_{ML}(\vec{y}) = \min_{\vec{x} \in \mathcal{C}} d_H(\vec{x}, \vec{y})$$

where, as usual, d_H represents the Hamming distance.

   (b) What is the maximum likelihood decoder when $p > 1/2$?

2. In this question, we will explore the role of the binary entropy function

$$H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$$

in elementary combinatorics. Note that $2^{-nH(x)} = x^{nx}(1-x)^{n(1-x)}$. Denote

$$B(\vec{0}, pn) = \{\vec{x} \in \{0,1\}^n : d_H(\vec{x}, \vec{0}) \leq pn)\}.$$

Thats is $B(\vec{0}, pn)$ is the Hamming ball of radius pn around $\vec{0}$. Let $Vol(n, pn) = |B(\vec{0}, pn)|$ denote the volume, that is the number of binary vectors with Hamming weight no bigger than pn. We will show that $Vol(n, pn)$ is approximately equal to $2^{nH(x)}$ in this problem.

   (a) Use the Stirling approximation to that $\binom{n}{pn} > 2^{nH(p)-o(n)}$, where $o(n)$ is a function that satifies $\lim_{n\to\infty} \frac{o(n)}{n} \to 0$. The Stirling approximation can be found on wikipedia,

   (b) Use your result from the previous part to show that $Vol(n, pn) > 2^{nH(p)-o(n)}$

   (c) Note that

$$\begin{aligned} 1 &= (p + (1-p))^n \\ &= \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \\ &\geq \sum_{i=0}^{pn} \binom{n}{i} p^i (1-p)^{n-i} \end{aligned}$$

Show for $p < 1/2$ that the right hand side of the final inequality is no smaller than $Vol(n, pn)2^{-nH(p)}$.

(d) Use your result from the previous part to show that $Vol(n, pn) \leq 2^{nH(p)}$

(e) If $\vec{x}$ is a sequence generated randomly with each co-ordinate generated independently with Bernoulli 1/2, show that the probability that $\vec{x} \in B(\vec{0}, pn)$ is at most $2^{-n(1-H(p))+o(n)}$.

3. Problem 1.4 in your book

4. *Bhattacharaya Bound:* Solve problem 1.9 in your book, parts 1-5 and part 7. For all parts, assume that the binary symmetric channel BSC(p) is used.

5. Problem 1.11 parts 1-4.

6. Problem 2.3

7. Problem 2.10