Homework 2: Due Feb 19th on Dropbox in Angel

Posted on Feb 9th

Collaboration policy: You are allowed to discuss homework problems with your classmates. However, you should think about the problems before discussing them, and you must write up your own solutions. You must acknowledge any collaboration by writing your collaborators names on the front page of the assignment.

Citation policy: Try to solve the problems without reading any published literature or websites, besides the textbook and notes from class. If, however, you do use a solution or part of a solution that you found in the literature or on the web, you must cite it. Furthermore, you must write up the solution in your own words.

Submission: Homeworks typeset in latex are encouraged. If you submit a hand-written homework, please ensure that it is written legibly. The homeworks must be uploaded in the dropbox provided on Angel.

Evaluation: 10 points per question for the first 4 questions. 5 points for the last question.

1. Problem 2.14

2. Problem 2.17 parts 1, 2, and 3. Solve the problem assuming $q = 2$.

3. Problem 4.2. (Recall that an MDS code is one which achieves the Singleton Bound)

4. Plot the Singleton bound, the asymptotic Hamming Bound $R \leq 1 - H_q(\delta/2)$ for increasing q, and the Plotkin bound $R \leq 1 - \frac{\delta}{1-1/q}$ for increasing q. You can get the formula for $H_q(x)$ from the textbook.

5. Appending two codes. For any two vectors $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$, the vector $\mathbf{x}|\mathbf{y}$ is a $2n$ length vector with $\mathbf{y}$ appended at the end of $\mathbf{x}$

   - Let $\mathcal{C}_1$ be a (n, k_1, d_1) linear code, and $\mathcal{C}_2$ be a (n, k_2, d_2) linear code. What are the parameters of the code
   $$\mathcal{C} = \{\mathbf{x}|\mathbf{y} : \mathbf{x} \in \mathcal{C}_1, \mathbf{y} \in \mathcal{C}_2\}$$
   - Is the code $\mathcal{C}$ linear? If so, explain a procedure of finding the generator matrix $\mathbf{G}$ based on the generator matrices $\mathbf{G}_1, \mathbf{G}_2$ of $\mathcal{C}_1, \mathcal{C}_2$.

6. A (n, k, d) linear code is said to be *locally recoverable* with locality of r if a single erasure can be recovered from r unerased co-ordinates. Equivalently, for every codeword symbol is a linear combination of r other codeword symbols. Locally recoverable codes are significantly important in cloud storage systems. We will generalize the Singleton bound for locally recoverable codes in this question.

   - Show that a linear code has locality r if and only the following property is true: For every co-ordinate $i \in \{1, 2, \ldots, n\}$, there is a row in the parity check matrix with at most $r + 1$ non-zero entries, such that the ith entry of the row is not zero. In particular, show that the existence of such a row will ensure that the co-ordinate i is locally recoverable.
   - *Puncturing a locally recoverable code:* Consider the parity check matrix $H = [-P^T \;\; I]$ of a locally recoverable code. Let a row of the parity check matrix have non-zero entries in co-ordinates $n, l_1, \ldots, l_r \in \{1, 2, \ldots, n\}$ and zero in the remaining $n-(r+1)$ entries. Generate a new parity check matrix H_1 with $(n-(r+1))$ columns by removing the columns in $n, l_1, l_2, \ldots, l_r$. If $k > r$, show that H_1 forms the parity check matrix of a locally recoverable code with parameters $(n-(r+1), k-r, d)$ and locality r. *Hint: Show that the rank of the parity check matrix H_1 is $n - k - 1$.*

- Use the result of part (2) in combination with the Singleton Bound to show that $d \leq n - k$.
- *Bonus Question:* Suppose we apply the puncturing operation of part (b) recursively $t = \lceil \frac{k}{r} \rceil - 1$ times to get a series of locally recoverable codes with parity check matrices $H_1, H_2, \ldots, H_t$. Use the Singleton Bound to show that $d \leq n - k - \lceil k/r \rceil + 2$.