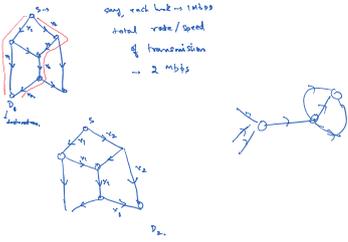
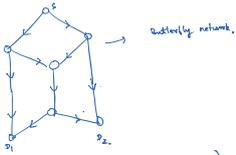


Network coding
 Directed Graph \rightarrow (V, E) edges
 set of vertices
 $E \subseteq V \times V$
 $s \in V$
 source node.
 $d \in V$
 destination node.



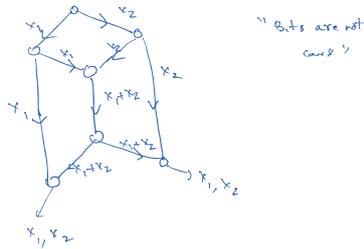
Max flow - Min cut theorem (consequence of Menger's theorem)
 In graph with unit capacity edges,
 max information flow = min-cut of graph.
 ↓
 smallest # of edges whose removal disconnects source and destination

strategy -> routing
Multicast communication
 source $\rightarrow s$
 destinations: D_1, D_2, \dots, D_n
 How fast can I communicate?



Obvious: $\max \text{flow} \leq \min_{D \in H} c(s, D)$
 max-flow \leq rate of slowest destination

with routing, $\text{flow} \leq 1.5$ (in example graph)
 Network coding \rightarrow flow = 2.



Network coding general result

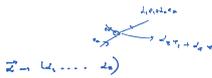
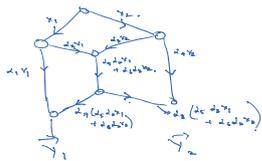
$$\max\text{-flow} = \min_c c(s, D_i)$$

MULTICAST THEOREM

proof idea

lemma 1 \rightarrow let $\vec{z} = (z_1, \dots, z_n)$
 $p(\vec{z}) \rightarrow$ non-zero multivariate poly in variables \vec{z} over field F
 then, over a sufficiently large extension of F
 $\exists \vec{z}_0$ s.t. $p(\vec{z}_0) \neq 0$
 \hookrightarrow evaluation
 e.g. $x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$
 $\neq \sqrt{2} \neq 0$

Lemma 2. Let H be a matrix of rank r .
 Then there is a $r \times r$ submatrix of H
 which is full-rank.



$$\vec{x} = [x_1 \ x_2]$$

$$y_1 = \vec{x}^T H_1(\vec{d}_0), \quad H_1 = \begin{bmatrix} d_1 & d_2 + d_3 \\ 0 & d_4 + d_5 \end{bmatrix}$$

Goal: find \vec{d}_0 s.t.

$$H_2 = \begin{bmatrix} d_6 & d_7 \\ d_8 & \dots \end{bmatrix}$$

$$\text{rank}(H_1(\vec{d}_0)) = 2 \quad \text{and} \\ \text{rank}(H_2(\vec{d}_0)) = 2$$

Two conditions above. Study them separately

Sub-goal 1
 \vec{d}_1 s.t. $\text{rank}(H_1(\vec{d}_1)) = 2$.

how? $\vec{d}_1 \rightarrow$ choose via ranking

$\Rightarrow \exists$ a 2×2 sub matrix H_1^* of H_1 which is full rank.

determinant of $H_1^(\vec{d}_1^*)$*

$$\Rightarrow \exists \text{ a determinant poly } D_1(\vec{d}_1^*) \text{ which is non-zero polynomial.}$$

Sub-goal 2

$$\vec{d}_2 \text{ s.t. } \text{rank}(H_2(\vec{d}_2)) = 2$$

how? $\vec{d}_2 \rightarrow$ ranking solution

$\Rightarrow \exists$ a 2×2 sub matrix H_2^* of H_2 which is full-rank

$$\Rightarrow \exists \text{ a determinant poly } D_2(\vec{d}_2^*) \text{ which is a non-zero polynomial}$$

$$\Rightarrow D_1(\vec{d}_1^*) D_2(\vec{d}_2^*) \text{ is a non-zero polynomial}$$

Lemma 1 \Rightarrow over sufficiently large extension

$$\exists \vec{d}_0 \text{ s.t. } D_1(\vec{d}_0^*) D_2(\vec{d}_0^*) \neq 0$$

$$\Rightarrow D_1(\vec{d}_0^*) \neq 0, \quad D_2(\vec{d}_0^*) \neq 0$$

$$\Rightarrow \text{rank}(H_1^*(\vec{d}_0^*)) = 2, \quad \text{rank}(H_2^*(\vec{d}_0^*)) = 2$$

$$\Rightarrow \text{rank}(H^*(\vec{d}_0^*)) \geq 2, \quad \text{rank}(H^*(\vec{d}_0^*)) \geq 2$$

Proof sketch for multicut theorem in general

Given a network with m destinations

$$D_1, \dots, D_m$$

$$\vec{y}_i = \vec{x}^T H_i(\vec{d}_i^*)$$

two variables

$\exists x$ integer of D_i

$$\text{Let } C = \min_{i=1, \dots, m} c(S - D_i) \text{ (min-cut b/w source } \leftarrow D_i)$$

$$\text{Goal: find } \vec{d}_0 \text{ s.t. } \text{rank}(H_i(\vec{d}_0^*)) = C.$$

Solution

m constraints. view them separately

$$\text{Find } \vec{d}_i \text{ s.t. } \text{rank}(H_i(\vec{d}_i^*)) \geq C$$

\rightarrow possible because of max-flow min-cut theorem!

$$\Rightarrow \exists \text{ a } C \times C \text{ sub matrix } H_i^*(\vec{d}_i^*) \text{ s.t. } \text{rank}(H_i^*(\vec{d}_i^*)) = C$$

$$\Rightarrow D_i(\vec{d}_i^*) \neq 0 \text{ (determinant poly)}$$

$$\Rightarrow D_i(\vec{d}_i^*) \text{ is not the zero poly, } i=1, \dots, m$$

$$\Rightarrow D_1(\vec{d}_1^*) D_2(\vec{d}_2^*) \dots D_m(\vec{d}_m^*) \text{ is not zero poly}$$

$$\Rightarrow \text{Lemma 1 for a sufficiently large extension, } \exists \vec{d}_0 \text{ s.t. } D_1(\vec{d}_0^*) D_2(\vec{d}_0^*) \dots D_m(\vec{d}_0^*) \neq 0$$

$$\Rightarrow D_1(z^{-1}) \neq 0, D_2(z^{-1}) \neq 0, \dots, D_m(z^{-1}) \neq 0$$

$$\Rightarrow \text{rk}(H_1^*(z^{-1})) = c, \text{rk}(H_2^*(z^{-1})) = c, \dots, \text{rk}(H_m^*(z^{-1})) = c$$

$$\Rightarrow \text{rank}(H_c(z^{-1})) \geq c \quad \forall z \quad \text{AED}$$

Last two topics in this course

- convolutional codes
 - LDPC codes
- } graph based codes

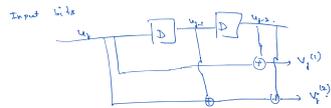
Convolutional codes

- Fast ML decoding possible (Viterbi's algo, Dynamic prog.)
- simple encoding

Example, inputs: u_1, u_2, \dots, u_{k-1}

$$v_1^{(1)} = u_1 + u_{k-2}$$

$$v_1^{(2)} = u_1 + u_{k-1} + u_{k-2}$$



Start with bits, ending bits $\rightarrow \infty$

Input $\rightarrow [1101]$

$$\text{outputs, } \left. \begin{array}{l} v^{(1)} \rightarrow [1111001] \\ v^{(2)} \rightarrow [100011] \end{array} \right\} \text{continued output } [111010000111]$$

k bits input, memory $k-1$ \rightarrow k bits loss (delay elements)

$$\text{Actual rate} = \frac{k}{2(k+1)}$$

$$\text{Design rate} = \frac{1}{2}$$

$$\text{In general, design rate} = \frac{1}{\sum_{i=1}^m q_i \text{ elements}}$$

Impulse response

$$g_1(z) = 1 + z^{-1} \quad \left| \begin{array}{l} c_1 = 1 \\ c_2 = 1 \end{array} \right.$$

$$g_2(z) = 1 + z^{-1} + z^{-2}$$

$$\text{b/p } v^{(1)}(z) = g_1(z) u(z) \quad \leftarrow u_1, u_2, \dots, u_{k-1}, z^{k-1}$$

$$v^{(2)}(z) = g_2(z) u(z)$$

Aside: polynomial multiplication \leftrightarrow discrete time convolution

$$\begin{aligned} (a_0 + a_1 z^{-1} + \dots + a_m z^{-m}) (b_0 + b_1 z^{-1} + \dots + b_n z^{-n}) &\iff [a_0 \ a_1 \ \dots \ a_m] \oplus [b_0 \ b_1 \ \dots \ b_n] \\ &\iff [c_0 \ c_1 \ \dots \ c_{m+n}] \end{aligned}$$