Optimal Sampling of Random Processes under Stochastic Energy Constraints

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***Abstract*—In this paper, we study the optimal sampling policy for an energy harvesting sensing system, which is designed to estimate a wide-sense stationary random process by using discrete-time samples collected by a sensor. The energy in the sensor is consumed by taking observations and is replenished randomly with energy harvested from the ambient environment. Our goal is to identify the optimal sampling policy that minimizes the estimation mean squared error (MSE) under stochastic energy constraints. The problem can be formulated as a stochastic programming problem, which is generally difficult to solve.**

We identify an asymptotically optimal solution to the problem by exploiting the properties of random processes with power-law decaying covariance. Specifically, with the help of a newly derived inverse covariance matrix of the random process, it is discovered that the linear minimum MSE (MMSE) estimation of the random process demonstrates a Markovian property. That is, the optimal estimation of any point in a time segment bounded by two consecutive samples can be achieved by using the knowledge of only the two bounding samples while ignoring all other samples. Such a Markovian property enables us to identify a lower bound of the long term average MSE. Motivated by the structure of the MSE lower bound, we then propose a simple best-effort sampling scheme by considering the stochastic energy constraints. It is shown that the best-effort sampling scheme is asymptotically optimal in the sense that, for almost every energy harvesting sample path, it achieves the MSE lower bound as time becomes large.

I. Introduction

Sensor networks equipped with energy harvesting devices have attracted great attentions recently. Compared with conventional sensor networks powered by batteries, the energy harvesting abilities of the sensor nodes make sustainable and environment-friendly sensor networks possible. The unique features of energy harvesting power supplies necessitate a completely different approach to the energy management in the communication or sensing systems.

One of the main challenges faced by the design of energy harvesting communication or sensing systems is the stochastic energy constraints imposed by the energy harvesting process. The amount of energy available in the system at a given time can be modeled as a random process due to the random energy arrivals. Many existing works on the design of energy harvesting communications employ an off-line deterministic optimization approach, which schedules data transmissions based on the knowledge of future energy arrivals [1]–[4]. The off-line scheduling methods treat future energy arrivals as a deterministic process even though the actual energy harvesting process is stochastic. Online scheduling methods address this problem by using only the statistics of energy arrivals and causal information available at the sensor. In [5], [6], sub-optimum on-line scheduling methods are proposed, and their performances are strictly worse than their off-line counterparts.

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In this paper, we consider the optimal design of an energy harvesting sensing system under stochastic energy constraints. The system monitors a time evolving physical quantity, such as temperature, humidity, or sunlight intensity, etc. We assume the physical quantity is a wide-sense stationary (WSS) random process within a time window. A sensor in the system samples the random process at discrete time intervals, and the continuous-time random process is estimated with discrete-time samples. It is assumed that each sampling operation consumes one unit of energy, and the random energy arrival is modeled as a Poisson process with parameter λ. The objective is to identify the optimal sampling scheme, i.e., the sequence of sampling instants, to minimize the time-averaged estimation mean squared error (MSE) under stochastic energy constraints. This is a stochastic optimization problem, and it is in general difficult to solve.

We focus on a special class of random processes with power-law decaying covariance, which enables us to find an asymptotically optimal solution to the problem. The power-law decaying covariance is observed in data collected from a variety of applications, such as the yields on an agricultural field [7], DNA sequences [8], condensed matter physics [9] [10], and long-range dependence of fractional Gaussian noise [11], etc. We first derive the explicit inverse of the covariance matrix, which is used to formulate the linear minimum MSE (MMSE) estimation of the random process. It is discovered that, with a power-law covariance, the MMSE estimation of a point on the random process using only the two adjacent discrete-time samples bounding the point yields exactly the same results as using all of the discrete-time samples. Therefore the random process exhibits a Markovian property in terms of MMSE estimation, in a sense that the estimation of each point requires only the knowledge of the two immediately adjacent samples. Such a Markovian property enables the identification of an analytical lower bound of the time-averaged MSE.

Motivated by the special structure of the MSE lower bound, we propose a simple best-effort uniform sampling scheme, where the sensor samples the random process at time instants $n/\lambda, n = 1, 2, \ldots$ as long as it has sufficient energy, and it remains silent otherwise. Such a sampling scheme does not require the knowledge of future energy arrivals. It is shown through both theoretical analysis and computer simulations that, for almost every energy harvesting sample path, the new best-effort uniform sampling scheme asymptotically achieves the MSE lower bound as the observation time becomes sufficiently large. Therefore, the best-effort sampling scheme is an asymptotically optimal solution to the stochastic optimization problem.

II. System Model and Problem Formulation

A. Energy Harvesting Model

Consider a sensor node powered by energy harvested from the ambient environment. It is assumed that the sensor node has an energy queue, such as a rechargeable battery or a super capacitor, to store the harvested energy. The energy queue is replenished randomly and consumed by taking observations. It is assumed that a unit amount of energy is required for one sensing operation. Since the harvested energy is usually very small compared to the battery capacity, it is assumed that the size of the energy queue is unlimited.

The energy arrival follows a Poisson process with parameter λ. Hence, energy arrivals occur in discrete time instants. Specifically, we use $t_1, t_2, \ldots, t_n, \ldots$ to represent the energy arrival epochs. Then, the energy inter-arrival times $t_i - t_{i-1}$ are exponentially distributed with means $1/\lambda$. Without loss of generality, it is assumed that the system starts with an empty energy queue at time 0.

A sampling policy or sensing scheduling policy is denoted as $\{l_n\}$, where l_n is the n-th sensing time instant. Since energy that has not arrived yet cannot be used at the current time, there is a causality constraint on the sampling policy. Specifically, we use $\sum_{n=1}^{\infty} \mathbf{1}_{t_n<t}$ and $\sum_{n=1}^{\infty} \mathbf{1}_{l_n \le t}$ to denote the total number of energy arrivals and sensing epochs up to time t, respectively. Here $\mathbf{1}_{\mathcal{E}}$ is an indicator function. It equals 1 if $\mathcal{E}$ is true, and equals 0 otherwise. Then, the energy causality constraint can be formulated as

$$\sum_{n=1}^{\infty} \mathbf{1}_{t_n<t} \geq \sum_{n=1}^{\infty} \mathbf{1}_{l_n \le t}, \quad \forall t > 0 \tag{1}$$

B. Sensing and Estimation Model

At each sensing epoch, the sensor collects a time-dependent physical quantity, $x(t)$, such as the temperature, humidity, or the pH value of the soil, etc. Due to the temporal redundancy of the monitored object, the data samples collected by the sensors are assumed to be correlated in the time domain. We assume $\{x(t)\}$ is a zero-mean WSS random process. The covariance between two data samples collected at l_m and l_n is assumed to satisfy the power-law decaying model as

$$\mathbb{E}\left[x(l_m)x(l_n)\right] = \rho^{|l_m - l_n|}, \tag{2}$$

where $\rho \in [0, 1]$ is the power-law coefficient. Define a vector containing the N data samples collected by the sensor as $\mathbf{x} = [x(l_1), \cdots, x(l_N)]^T \in \mathbb{R}^{N\times 1}$.

The sensing system attempts to reconstruct the continuous-time random process by using discrete-time samples collected by the sensors. The linear MMSE estimate of the random process at an arbitrary time t can be expressed as

$$\hat{x}(t) = \mathbf{r}_t^T \mathbf{R}_{xx}^{-1} \mathbf{x} \tag{3}$$

$\mathbf{r}_t = \mathbb{E}[x(t)\mathbf{x}^T] \in \mathbb{R}^{N\times 1}$, and $\mathbf{R}_{xx} = \mathbb{E}\left[\mathbf{x}\mathbf{x}^T\right] \in \mathbb{R}^{N\times N}$. The corresponding MSE for the reconstructed signal at time t is

$$\sigma^2(t) = \mathbb{E}\left[|\hat{x}(t) - x(t)|^2\right] = 1 - \mathbf{r}_t^T \mathbf{R}_{xx}^{-1} \mathbf{r}_t \tag{4}$$

where $\hat{x}(t)$ is an estimate of $x(t)$.

C. Problem Formulation

Our objective is to design an online sensing policy $\{l_n\}$, such that the expected long-term average MSE is minimized, subject to the energy constraint at every time instant. The optimization problem is formulated as

$$\min_{\{l_n\}} \quad \limsup_{T\to+\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \sigma^2(t)dt\right] \tag{5}$$

$$\text{s.t.} \quad \sum_{n=1}^{\infty} \mathbf{1}_{t_n<t} \geq \sum_{n=1}^{\infty} \mathbf{1}_{l_n \le t}, \quad \forall t > 0 \tag{6}$$

where the expectation in the objective function is taken over all possible energy harvesting sample paths. This is a stochastic optimization problem, and in general is hard to obtain a closed-form optimal solution. Moreover, the complex form of the objective function makes the problem even more complicated. In the following sections, we will show that the objective function is analytically tractable with a power-law covariance, and identify the properties of the corresponding optimal solution. The properties motivate a best-effort uniform sampling strategy, which is then proven to be asymptotically optimal as $T \to \infty$.

III. Statistical Properties of MMSE Estimation

In this section, we study the statistical properties of the random process obtained from the linear MMSE estimation. The analysis is enabled by an explicit expression of the inverse covariance matrix with power-law elements. Results obtained in this section will be used to identify the optimum sensing policy in an energy harvesting sensing system in the next section.

A. MMSE Estimation

Theorem 1 *Consider the linear MMSE estimation of $x(t)$ by using the N discrete samples $\{x(l_k)\}_{k=1}^N$ with $l_1 < l_2 \cdots < l_N$, as described in* (3).

If $l_n < t < l_{n+1}$ for $n = 1, \cdots, N-1$, then

$$\hat{x}(t) = \frac{\rho^{t-l_n} - \rho^{-(t-l_n)} a_{n+1}^2}{1 - a_{n+1}^2} \times x(l_n) + \frac{\rho^{l_{n+1}-t} - \rho^{-(l_{n+1}-t)} a_{n+1}^2}{1 - a_{n+1}^2} \times x(l_{n+1}) \tag{7}$$

where $a_n = \rho^{l_n - l_{n-1}}$ *for* $n = 2, \cdots, N$, *and* $a_1 = a_{N+1} = 0$.

If $t < l_1$, *then*

$$\hat{x}(t) = \rho^{l_1 - t} \times x(l_1) \tag{8}$$

If $t > l_N$, *then*

$$\hat{x}(t) = \rho^{t - l_N} \times x(l_N) \tag{9}$$

Due to space limitations, the proof of Theorem 1, as well as the proofs of Corollaries 1, 2 and 3, Lemmas 1, 2 and Theorem 3, are omitted in this paper. All proofs can be found in [12].

The results in Theorem 1 state that the MMSE estimation of the random process with a power-law decaying covariance has a Markovian behavior, that is, the estimation of $x(t)$ with $l_n < t < l_{n+1}$ only requires the knowledge of $x(l_n)$ and $x(l_{n+1})$, the two closest samples bounding t. The N-dimensional estimation problem is thus reduced to a two-dimensional estimation problem with a fixed complexity. We have the following corollary regarding a closed-form expression of the MSE $\sigma^2(t)$.

Corollary 1 *For the MMSE estimator described in Theorem 1, the MSE,* $\sigma^2(t) = \mathbb{E}\left[|\hat{x}(t) - x(t)|^2\right]$, *is*

$$\sigma^2(t) = \frac{1 + a_{n+1}^2 - \rho^{2t - 2l_n} - \rho^{2l_{n+1} - 2t}}{1 - a_{n+1}^2} \tag{10}$$

where $l_n \leq t \leq l_{n+1}$ *for* $n = 0, \cdots, N$, *with* $l_0 = -\infty$ *and* $l_{N+1} = \infty$, *and* $a_n = \rho^{l_n - l_{n-1}}$ *for* $n = 2, \cdots, N$ *and* $a_1 = a_{N+1} = 0$.

Corollary 1 indicates that for $l_n \leq t \leq l_{n+1}$, $\sigma^2(t)$ only depends on two sensing instants l_n and l_{n+1}. This implies that the objective function in (5) can be decomposed into a summation of integrals, where each integral covers one interval $[l_n, l_{n+1}]$, for $n = 0, \cdots, N$.

Corollary 2 *Let* $f_{n+1} = \int_{l_n}^{l_{n+1}} \sigma^2(t)dt$ *with* $\sigma^2(t)$ *given in (10) for* $n = 1, 2, \ldots, N-1$. *Then,* f_{n+1} *is a function of* $l_{n+1} - l_n := d_{n+1}$, *denoted as* $f(d_{n+1})$, *with*

$$f(d) = d\frac{1 + \rho^{2d}}{1 - \rho^{2d}} + \frac{1}{\log \rho} \tag{11}$$

The function $f(d)$ *has the following properties*

1) $f(0) = 0$.
2) $f(d)$ *is monotonically increasing in* d.
3) $f(d)$ *is a convex function for* $d \geq 0$.

Corollary 3 *Let* $f_1 = \int_0^{l_1} \sigma^2(t)dt$ *and* $f_{N+1} = \int_{l_N}^{T} \sigma^2(t)dt$ *with* $\sigma^2(t)$ *given in (10). Then* $f_1 = g(d_1)$ *and* $f_{N+1} = g(d_{N+1})$, *with*

$$g(d) = d + \frac{1}{2\log\rho}\left(1 - \rho^{2d}\right) \tag{12}$$

Moreover, we have $g(d) \geq f(d)$.

Corollaries 2 and 3 state that if N samples are taken over $(0, T)$, then, for any duration bounded by two consecutive sampling epochs, the total MSE for the estimation over that duration has a common structure, which is a function of the length of the duration. The estimation MSE over $[0, l_1]$ and $[l_n, T]$ has a different form, due to the fact that they have only one bounding sampling epoch.

From Corollaries 2 and 3, the time-averaged MSE in the cost function in (5) can be written as

$$\begin{aligned}\frac{1}{T}\int_0^T \sigma^2(t)dt &= \frac{1}{T}\left[g(d_1) + \sum_{n=2}^{N} f(d_n) + g(d_{N+1})\right], \\ &\geq \frac{1}{T}\sum_{n=1}^{N+1} f(d_n)\end{aligned} \tag{13}$$

where the time-averaged MSE is decomposed as the summation of a sequence of increasing and convex functions of intervals bounded by two consecutive discrete-time samples. As $T \to \infty$, it is expected that the boundary functions $g(d_1)$ and $g(d_{N+1})$ will have negligible impacts on the average MSE.

IV. An Asymptotically Optimal Sampling Policy under Stochastic Energy Constraints

The optimization problem in (5) is stochastic and in general hard to solve. However, with Corollaries 2 and 3, we first show that the optimal solution has a lower bound, which corresponds to a uniform sampling policy with a fixed sampling rate. Motivated by this observation, we then propose a uniform best-effort sampling policy and prove its optimality by showing that the uniform best-effort policy asymptotically achieves the lower bound.

A. A Lower Bound of the Average MSE

Definition 1 *A sampling policy* $\{l_n\}$ *is feasible if* $E(l_n^-) \geq 1$, $\forall n \geq 1$, *where* $E(t)$ *is the energy available in the energy queue at time* t, *and* l_n^- *is the time instant right before* l_n.

Lemma 1 *Under every feasible scheduling policy, we have*

$$\limsup_{T \to +\infty} \frac{N_T}{T} \leq \lambda, \quad a.s. \quad \forall i \tag{14}$$

where $N_T = \sum_{n=1}^{\infty} \mathbf{1}_{l_n \leq T}$ *is the total number of samples taken in* $[0, T]$.

Based on the properties of function $f(d)$ stated in Corollary 2 and Lemma 1, we establish the following lower bound on the objective function.

Lemma 2 *The objective function in (5) is lower bounded as*

$$\min_{\{l_n\}} \limsup_{T \to +\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \sigma^2(t)dt\right] \geq \lambda f\left(1/\lambda\right) \tag{15}$$

B. A Best-effort Sampling Policy

Motivated by the lower bound in Lemma 2, we propose a best-effort sampling policy that asymptotically achieves the MSE lower bound as $T \to \infty$. During the derivation of the lower bound, we note that the equality can be achieved if $d_1 = d_2 = \cdots = \frac{1}{\lambda}$. Therefore, the lower bound can be achieved if uniform sampling with a sampling period $\frac{1}{\lambda}$ is employed. However, due to the stochastic nature of the energy sources, uniform sampling is in general infeasible in energy harvesting sensing systems. We thus propose a best-effort uniform sampling policy described as follows.

Definition 2 (Best-effort Sampling Policy) *The sensor is scheduled to perform the sensing task at $l_n = n/\lambda$. The sensor performs the sensing task at l_n if $E(l_n^-) \geq 1$; Otherwise, the sensor keeps silent until the next sensing epoch.*

The best-effort uniform sampling policy is always feasible because it only samples the random process if there is sufficient energy left in the energy queue. With the proposed best-effort sampling policy, the sensor attempts to sample the random process at uniform intervals, but will only do so if there is sufficient energy for the sampling operations. As a result, the interval bounded by two consecutive samples varies due to the stochastic availability of the energy sources, and the interval is always an integer multiple of $1/\lambda$.

We will show next that the proposed best-effort sampling policy can asymptotically achieve the MSE lower bound given in Lemma 2, which states the best possible MSE achievable for all possible sampling policies.

Theorem 2 *Under the best-effort uniform sampling policy, we have*

$$\lim_{T\to+\infty} \frac{N_T}{T} = \lambda \quad a.s.$$

The proof of Theorem 2 is provided in the Appendix.

Theorem 3 *The uniform sampling policy $\{l_n\}$ is optimal, i.e.,*

$$\limsup_{T\to+\infty} \left[\frac{1}{T} \int_0^T \sigma^2(t)dt \right] = \lambda f(1/\lambda) \quad a.s.$$

Theorem 2 indicates that the best-effort sampling scheme is asymptotically feasible almost surely, i.e., the sensor has sufficient energy to perform the task for almost every scheduled sampling epoch. Theorem 3 indicates that for almost every energy harvesting sample path, the average estimation MSE given by the MMSE estimator with samples collected according to the best-effort sampling scheme converges to the lower bound. Therefore, the best-effort sampling policy is an asymptotically optimal solution to the stochastic optimization problem in (5).

V. Numerical and Simulation Results

Numerical and simulation results are provided in this section to demonstrate the performance of the best-effort sampling policy.

Fig. 1 plots the graph of the function $f(d)$, which is the MSE integrated over the time duration between two consecutive sensing epochs. As shown in Corollary 2, $f(d)$ is an increasing and convex function in d, and $f(0) = 0$. A larger ρ yields a smaller $f(d)$ due to the stronger correlation between the two samples.

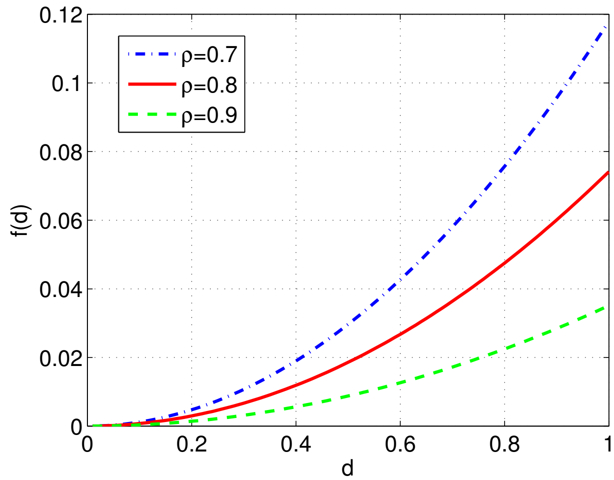

Fig. 1: Numerical evaluation of $f(d)$ as a function of the distance between two consecutive sensing epochs.

Next, we evaluate the performance of the proposed best-effort sampling policy through simulations in Fig. 2. The energy harvesting rates are set to be $\lambda = 1, 2, 3$ per unit time, respectively. For each λ, 1,000 energy harvesting profiles are generated according to the Poisson distribution, and the best-effort sampling is performed for each energy harvesting profile. The sensing rate, N_T/T, for each energy harvesting profile is tracked and recorded. The average sensing rate for the 1,000 sample paths is plotted as a function of T in Fig. 2. It is observed that the average sensing rate approaches λ asymptotically as T increases, as predicted in Theorem 2. Thus the best-effort sampling policy asymptotically approaches the behavior of uniform sampling when $T > 400$.

The simulation MSE obtained by employing the best-effort sampling policy is shown in Figs. 3 and 4 for various values of λ and ρ. The MSE is averaged over $1,000$ independently generated energy harvesting profiles. In Fig. 3, the value ρ is fixed at 0.9. In Fig. 4, $\lambda = 1$. It is observed from both figures that the MSE curves gradually approach their respective lower bounds $\lambda f(1/\lambda)$ as T increases. When $T = 500$, there is only a very small difference between the simulation MSE and the analytical lower bound. The results indicate that the proposed best-effort sampling policy is asymptotically optimal.

In addition, in Fig. 3, the MSE is a decreasing function in λ. This indicates that the sensing performance strictly improves when the energy harvesting rate increases, which is intuitive since more sensing samples can be collected per unit time on average. In Fig. 4, the MSE is a decreasing function in ρ. This is due to the fact that a stronger correlation results in a better estimation.

VI. Conclusions

The optimal sampling policy of a WSS random process with a power-law decaying covariance function was studied

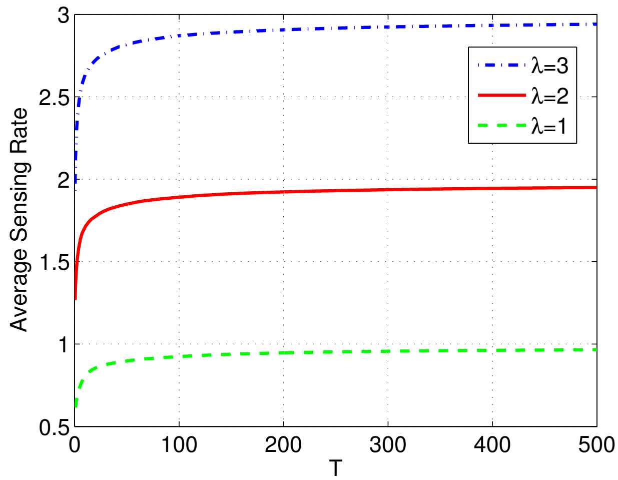

Fig. 2: Average sensing rate as a function of T.

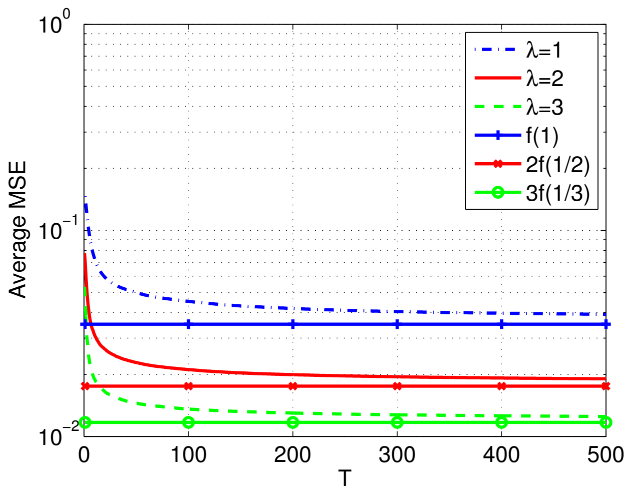

Fig. 3: Average sensing rate as a function of T ($\rho = 0.9$).

for a sensing system powered by energy harvesting devices. There are two major contributions of this work. First, we explicitly identified the optimal linear MMSE estimation of a random process with power-law decaying covariance, and showed that the optimal estimation of any point in an interval bounded by two consecutive samples relies only on the two bounding samples of the interval. Second, we presented an asymptotic optimal sampling policy that minimizes the time-average estimation MSE under stochastic energy constraints. The sampling policy was motivated by the structure of the MSE lower bound, which was derived by using the Markovian property of the MMSE estimation. Even though the asymptotic results require the sensing time $T \to \infty$, simulation results demonstrated that the performance of the proposed best-effort sampling policy approaches the MSE lower bound when $T \geq 500$.

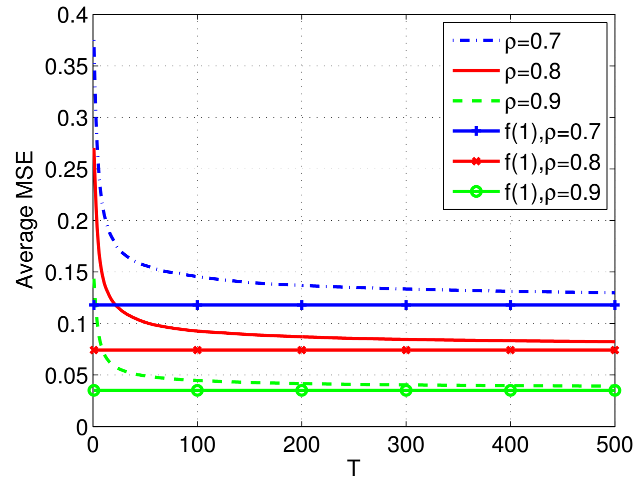

Fig. 4: Average sensing rate as a function of T ($\lambda = 1$).

APPENDIX
PROOF OF THEOREM 2

The uniform best-effort sampling policy partitions the time axis into slots, each with length $1/\lambda$. Consider the number of energy arrivals during a slot, denoted as A. Due to the Poisson process assumption of the energy arrival process, we have

$$\mathbb{P}\left[A = k\right] = \frac{e^{-1}}{k!}, \quad k = 0, 1, 2 \ldots$$

Let $E(n/\lambda)$ be the energy level of the sensor right before the scheduled sensing epoch n/λ. Based on $E(n/\lambda)$, we can group the time slots into segments with lengths $u_0, v_1, u_1, \ldots, v_k, u_k, \ldots$, where u_is correspond to the segments when $E(n/\lambda) = 0$ and v_is correspond to the segments when $E(n/\lambda) > 0$, as shown in Fig. 5. E jumps from zero to some positive value e_i at the end of the segment corresponding to u_i. Therefore, u_i follows an independent geometric distribution

$$\mathbb{P}\left[u_i = \frac{k}{\lambda}\right] = e^{-(k-1)}(1 - e^{-1}), \quad k = 1, 2 \ldots$$

and v_i follows a "random walk" with increment $A - 1$ starting at some positive level e_i until it hits 0. Note that v_i contains a random walk Γ_i which starts at e_i and finishes at $e_i - 1$ for the first time. Denote the duration of Γ_i as τ_i.

Let K_T be the number of segments with $E(n/\lambda) = 0$ during T. Note that $T = N_T/\lambda + \sum_{i=0}^{K_T} u_i$. Therefore, to show $N_T/T \to \lambda$ almost surely, it suffices to show that

$$\lim_{T \to \infty} \frac{\sum_{i=0}^{K_T} u_i}{T} = 0, \quad a.s.$$

Note that

$$\frac{\sum_{i=0}^{K_T} u_i}{T} = \frac{\sum_{i=0}^{K_T} u_i}{K_T} \frac{K_T}{T} \leq \frac{\sum_{i=0}^{K_T} u_i}{K_T} \frac{K_T}{\sum_{i=1}^{K_T} \tau_i}$$

Therefore, if we can prove that $\mathbb{P}\left[v_i < \infty\right] = 1$, then $K_T \to$

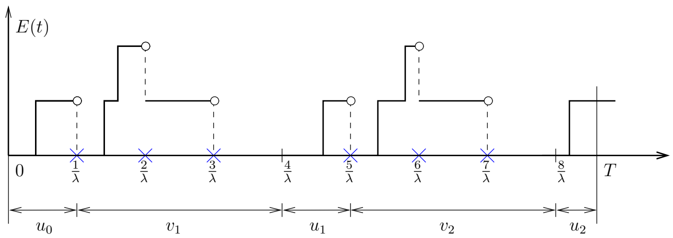

Fig. 5: An energy level evolution sample path. Crosses represent actual sensing epochs.

∞ as $T \to \infty$. In addition, by the strong law of large numbers,

$$\lim_{T\to\infty} \frac{\sum_{i=0}^{K_T} u_i}{K_T} = \frac{1}{\lambda(1-e^{-1})}, \quad a.s.$$

Then, to prove Theorem 2, it suffices to show that

$$\lim_{T\to\infty} \frac{K_T}{\sum_{i=1}^{K_T} \tau_i} = 0, \quad a.s. \tag{16}$$

In the following, we will first prove $\mathbb{P}[v_i < \infty] = 1$, and then show (16) holds.

Consider a "random walk" $\{\Omega_k\}_{k=0}^{\infty}$, which starts with 1 and increments with $A-1$. Denote the first 0-hitting time for $\{\Omega_k\}_{k=0}^{\infty}$ as κ. Then, $\Omega_0 = 1, \Omega_\kappa = 0$.

Define a Martingale process $\{\exp(-\alpha\Omega_k - \gamma(\alpha)k)\}_{k=0}^{\infty}$ with $\alpha > 0$ and $\gamma(\alpha) = \lambda(e^{-\alpha} - (1-\alpha)) > 0$. Based on the property of a Martingale, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp(-\alpha\Omega_k - \gamma(\alpha)k)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\exp(-\alpha\Omega_k - \gamma(\alpha)k)|\Omega_1, \ldots, \Omega_{k-1}\right]\right] \\
&= \mathbb{E}\left[\exp(-\alpha\Omega_{k-1} - \gamma(\alpha)(k-1))\right]
\end{aligned}$$

Applying this equality recursively, we have

$$\begin{aligned}
\exp(-\alpha) &= \mathbb{E}\left[\exp(-\alpha\Omega_\kappa - \gamma(\alpha)\kappa)\right] && (17) \\
&= \mathbb{E}\left[(\mathbf{1}_{\kappa<\infty} + \mathbf{1}_{\kappa=\infty}) \cdot \exp(-\alpha\Omega_\kappa - \gamma(\alpha)\kappa)\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\kappa<\infty} \cdot \exp(-\alpha\Omega_\kappa - \gamma(\alpha)\kappa)\right] && (18)
\end{aligned}$$

where the equality in (18) holds due to the fact that $\exp(-\gamma(\alpha)\cdot\infty) = 0$. Let $\alpha \to 0^+$, then $\gamma(\alpha) \to 0^+$, and the equation becomes

$$1 = \mathbb{E}\left[\mathbf{1}_{\kappa<\infty}\right] = \mathbb{P}\left[\kappa < \infty\right]$$

i.e., the probability of hitting 0 in finite time is 1.

Similarly, we can prove that starting with any $e_i > 0$, the probability that the first 0-hitting time is finite equals 1, i.e., $\mathbb{P}[v_i < \infty] = 1$.

Since $\Omega_\kappa = 0$, (17) is equivalent to

$$\mathbb{E}\left[\exp(-\gamma(\alpha)\kappa)\right] = \exp(-\alpha).$$

We note that by shifting Γ_i to initial state 1, it virtually follows the same random walk $\{\Omega_k\}_k$. For such K_T i.i.d random walks with 0-hitting times τ_i, we have

$$\mathbb{E}\left[\exp\left(-\gamma(\alpha)\left(\sum_{i=1}^{K_T}\tau_i\right)\right)\right] = \exp(-K_T\alpha), \tag{19}$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left[\frac{K_T}{\sum_{i=1}^{K_T}\tau_i} > \epsilon\right] = \mathbb{P}\left[\sum_{i=1}^{K_T}\tau_i < \frac{K_T}{\epsilon}\right] \\
&= \mathbb{P}\left[\exp\left(-\gamma(\alpha)\left(\sum_{i=1}^{K_T}\tau_i\right)\right) > \exp\left(-\gamma(\alpha)\frac{K_T}{\epsilon}\right)\right] && (20) \\
&\le \frac{\exp(-K_T\alpha)}{\exp(-\gamma(\alpha)\frac{K_T}{\epsilon})} = \exp\left(-K_T\left(\alpha - \frac{\gamma(\alpha)}{\epsilon}\right)\right) && (21)
\end{aligned}$$

where (20) follows from the monotonicity of e^{-x} and (19), and (21) follows from Markov's inequality.

Since $\gamma(\alpha) = O(\alpha^2)$, for any $\epsilon > 0$, we can always find a α to have $\alpha - \frac{\gamma(\alpha)}{\epsilon} > 0$, and then the probability decays exponentially in K_T. According to Borel-Cantelli lemma [13], we have (16) hold, which completes the proof.

References

[1] J. Yang and S. Ulukus, "Optimal packet scheduling in an energy harvesting communication system," *IEEE Trans. Comm.*, vol. 60, no. 1, pp. 220–230, January 2012.

[2] M. A. Antepli, E. Uysal-Biyikoglu, and H. Erkal, "Optimal packet scheduling on an energy harvesting broadcast link," *IEEE Jour. on Selected Areas in Commun.*, vol. 29, no. 8, pp. 1721–1731, September 2011.

[3] J. Yang, O. Ozel, and S. Ulukus, "Broadcasting with an energy harvesting rechargeable transmitter," *IEEE Trans. Wireless Comm.*, vol. 11, no. 2, pp. 571–583, February 2012.

[4] J. Yang and S. Ulukus, "Optimal packet scheduling in a multiple access channel with energy harvesting transmitters," *Journal of Communications and Networks*, vol. 14, pp. 140–150, April 2012.

[5] O. Ozel, K. Tutuncuoglu, J. Yang, S. Ulukus, and A. Yener, "Transmission with energy harvesting nodes in fading wireless channels: Optimal policies," *IEEE Jour. on Selected Areas in Commun.*, vol. 29, no. 8, pp. 1732–1743, September 2011.

[6] K. Tutuncuoglu and A. Yener, "Sum-rate optimal power policies for energy harvesting transmitters in an interference channel," *Journal of Communications and Networks*, vol. 14, no. 2, pp. 151–161, April 2012.

[7] P. Whittle, "Topographic correlation, power-law covariance functions, and diffusion," *Biometrika*, vol. 49, no. 3/4, pp. 305–314, Dec., 1962.

[8] S. Buldyrev, "Power law correlations in dna sequences," *Madame Curie Bioscience Database [Internet], Available from: http://www.ncbi.nlm.nih.gov/books/NBK6262/*, 2000-.

[9] G. Roumpos, M. Lohse, W. H. Nitsche, J. Keeling, M. H. Szymanska, P. B. Littlewood, A. Loffler, S. Hofling, L. Worschech, A. Forchel, and Y. Yamamoto, "Power-law decay of the spatial correlation function in exciton-polariton condensates," *PNAS*, vol. 109, pp. 6467–6472, 2012.

[10] H. Stanley, S. Buldyrev, A. Goldberger, S. Havlin, C.-K. Peng, and M. Simons, "Long-range power-law correlations in condensed matter physics and biophysics," *Physica A: Statistical Mechanics and its Applications*, vol. 200, no. 1-4, pp. 4 – 24, 1993. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0378437193905004

[11] T. Gneiting, "Power-law correlations, related models for long-range dependence and their simulation," *Journal of Applied Probability,*, vol. 37, pp. 1104–1109, 2000.

[12] J. Yang and J. Wu, "Optimal sampling of random processes in energy harvesting sensing systems," *IEEE Jour. on Selected Areas in Commun.*, submitted March 2014.

[13] A. Klenke, *Probability Theory: A Comprehensive Course (Universitext)*. Springer, 2007.