A Non-Asymptotic Achievable Rate for the AWGN Energy-Harvesting Channel using Save-and-Transmit

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***Abstract*—This paper investigates the information-theoretic limits of the additive white Gaussian noise (AWGN) energy-harvesting (EH) channel in the finite blocklength regime. The EH process is characterized by a sequence of i.i.d. random variables with finite variances. We use the save-and-transmit strategy proposed by Ozel and Ulukus (2012) together with Shannon's non-asymptotic achievability bound to obtain a lower bound on the achievable rate for the AWGN EH channel. The first-order term of the lower bound on the achievable rate is equal to C and the second-order (backoff from capacity) term is proportional to $-\sqrt{\frac{\log n}{n}}$, where n denotes the blocklength and C denotes the capacity of the EH channel, which is the same as the capacity without the EH constraints. The constant of proportionality of the backoff term is found and qualitative interpretations are provided.**

I. Introduction

The energy-harvesting (EH) channel consists of one source equipped with an energy buffer, and one destination. For simplicity, in this paper, we assume that the buffer has infinite capacity. At each discrete time $k \in \{1, 2, \ldots\}$, a random amount of energy $E_k \in [0, \infty)$ arrives at the buffer and the source transmits a symbol $X_k \in (-\infty, \infty)$ such that

$$\sum_{\ell=1}^{k} X_\ell^2 \le \sum_{\ell=1}^{k} E_\ell \qquad \text{almost surely.} \tag{1}$$

In other words, the total harvested energy $\sum_{\ell=1}^{k} E_\ell$ must be no smaller than the energy of the codeword $\sum_{\ell=1}^{k} X_\ell^2$ at every discrete time k for transmission to take place successfully. We assume that $\{E_\ell\}_{\ell=1}^{\infty}$ are independent and identically distributed (i.i.d.) non-negative random variables, where $\mathrm{E}[E_1] = P$ and $\mathrm{E}[E_1^2] < +\infty$. The destination receives $Y_k = X_k + Z_k$ at time slot k for each $k \in \mathbb{N}$ where $\{Z_k\}_{k=1}^{\infty}$ are i.i.d. standard normal random variables. We refer to the above EH channel as the *additive white Gaussian noise (AWGN) EH channel*. It was shown by Ozel and Ulukus [1] that the capacity of the AWGN EH channel is

$$C \triangleq \frac{1}{2}\log(1+P), \tag{2}$$

where $P = \mathrm{E}[E_1]$ is the expectation of the amount of harvested energy for each energy arrival. The AWGN EH channel models real-world, practical situations where energy may not be fully available at the time of transmission and its unavailability may result in the transmitter not being able to put out the desired codeword. This model is applicable in large-scale sensor networks where each node is equipped with an EH device that collects a stochastic amount of energy. See [2] for a comprehensive review of recent advances in EH wireless communications.

Although the capacity in (2) is unchanged vis-à-vis the AWGN channel *without* the EH constraints, we show in this work that if one uses the save-and-transmit strategy [1] to take the EH constraints into account, then there can potentially be a significant backoff from capacity at moderate blocklengths compared to the case when EH constraints are absent (cf. [3]).

A. Main Contribution

We prove an achievable finite blocklength bound for the AWGN EH channel under the EH constraints in (1) based on the save-and-transmit strategy of [1]. During the saving phase of the save-and-transmit strategy, we save energy for a certain number of time slots and no information is transmitted. Subsequently, during the transmission phase, we use the remaining time slots to send information. By carefully developing various concentration bounds to control the probability that the available energy is insufficient to support the transmitted codeword during the transmission phase (i.e., that $\sum_{\ell=1}^{k} E_\ell < \sum_{\ell=1}^{k} X_\ell^2$), we show that the backoff from capacity C at a blocklength n is no larger than $O(\sqrt{n^{-1}\log n})$. Furthermore, by scrutinizing the analysis of Ozel and Ulukus [1], one can also deduce that the backoff from capacity is no larger than $O(n^{-1/2}\log n)$. Thus, our analysis results in a slightly smaller (tighter) backoff than what was implied by the authors in [1].

In addition, our analysis only requires minimal statistical assumptions on the EH process $\{E_\ell\}_{\ell=1}^{\infty}$. Indeed, apart from assuming that the process is i.i.d., we only assume that the second moment of the EH random variable E_1 is bounded, i.e., $\mathrm{E}[E_1^2] < \infty$. In previous results such as [1, Lemmas 1 & 2], more restrictive assumptions on E_1 were made (e.g., $\mathrm{E}\big[e^{E_1^\gamma}\big]$ is bounded for some $\gamma \in (0,1)$), which may be hard to verify in practice.

B. Related Work

Information-theoretic characterizations of EH communication channels have been investigated recently. As energy arrives randomly to the transmitter, codewords must satisfy the cumulative stochastic energy constraints. The impact of the stochastic energy supply on the channel capacity was

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characterized for the AWGN channel with an i.i.d. EH process in [1] and with a stationary ergodic EH process in [4]. The aforementioned studies showed that with an unlimited battery, the capacity of the AWGN channel with stochastic energy constraints is equal to the capacity of the same channel under an average power constraint, as long as the average power equals the average recharge rate of the battery.

The study of finite blocklength fundamental limits in Shannon-theoretic problems was undertaken by Polyanskiy, Poor and Verdú [3]. Such a study is useful as it provides guidelines regarding the required backoff from the asymptotic fundamental limit (capacity) when one operates at finite blocklengths. For a survey, please see [5].

C. Paper Outline

This paper is organized as follows. The notation used in this paper is described in the next subsection. Section II states the formulation of the AWGN EH channel and presents our main theorem. Numerical results are also provided. Section III describes the save-and-transmit strategy and presents a proof sketch of our main theorem.

D. Notation

We use X^n to denote the random tuple $(X_1, X_2, \ldots, X_n)$. We let $p_{Y|X}$ denote the conditional probability distribution of Y given X, and let $p_{X,Y} = p_X p_{Y|X}$ denote the probability distribution of (X, Y). To make the dependence on the distribution explicit, we let $\Pr_{p_X}\{g(X) \in \mathcal{A}\}$ denote $\int_{x\in\mathcal{X}} p_X(x)\mathbf{1}\{g(x) \in \mathcal{A}\}\,\mathrm{d}x$ for any set $\mathcal{A} \subseteq \mathbb{R}$ and any real-valued function g with domain $\mathcal{X}$. The expectation and the variance of $g(X)$ are denoted as $\mathrm{E}_{p_X}[g(X)]$ and $\mathrm{Var}_{p_X}[g(X)]$ respectively. We let $\mathcal{N}(z;\mu,\sigma^2) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}}\exp\big(-\frac{(z-\mu)^2}{2\sigma^2}\big)$ denote the probability density function of the Gaussian random variable Z whose mean and variance are μ and σ^2 respectively. We will take all logarithms to base e throughout this paper.

II. AWGN Energy-Harvesting Channel

A. Problem Formulation and Main Result

The AWGN EH channel consists of one source and one destination, denoted by s and d respectively. Node s transmits information to node d in n time slots as follows. Node s chooses message W and sends W to node d, where W is uniform on its alphabet. Then for each $k \in \{1, 2, \ldots, n\}$, node s transmits $X_k \in \mathbb{R}$ and node d receives $Y_k \in \mathbb{R}$ in time slot k. Let $E_1, E_2, \ldots, E_n$ be i.i.d. random variables that satisfy $\Pr\{E_1 < 0\} = 0$, $\mathrm{E}[E_1] = P$ and $\mathrm{E}[E_1^2] < \infty$. We assume the following for each $k \in \{1, 2, \ldots, n\}$:

(i) E_k and $(W, E^{k-1}, X^{k-1}, Y^{k-1})$ are independent, i.e.,

$$p_{W,E^k,X^{k-1},Y^{k-1}} = p_{E_k} p_{W,E^{k-1},X^{k-1},Y^{k-1}}$$

(ii) Every codeword X^n transmitted by s should satisfy

$$\Pr\left\{\sum_{\ell=1}^k X_\ell^2 \le \sum_{\ell=1}^k E_\ell\right\} = 1. \tag{3}$$

After n time slots, node d declares $\hat{W}$ to be the transmitted W based on Y^n. Formally, we define a code as follows:

Definition 1: An (n, M)*-code* consists of a message set $\mathcal{W} \triangleq \{1, 2, \ldots, M\}$ at node s where W is uniform on $\mathcal{W}$, a sequence of encoding functions $f_k : \mathcal{W} \times \mathbb{R}_+^k \to \mathbb{R}$ for each $k \in \{1, 2, \ldots, n\}$ such that $X_k = f_k(W, E^k)$ and (3) holds, and a decoding function $\varphi : \mathbb{R}^n \to \mathcal{W}$ at node d which produces $\hat{W} = \varphi(Y^n)$.

Definition 2: The *AWGN EH channel* is characterized by $q_{Y|X}$ such that the following holds for any (n, M)-code: For each $k \in \{1, 2, \ldots, n\}$,

$$p_{W,E^k,X^k,Y^k} = p_{W,E^k,X^k,Y^{k-1}} p_{Y_k|X_k}$$

where

$$p_{Y_k|X_k}(y_k|x_k) = q_{Y|X}(y_k|x_k) = \mathcal{N}(y_k - x_k; 0, 1). \tag{4}$$

We call an (n, M)-code with average probability of decoding error $\Pr\{\hat{W} \neq W\}$ no larger than ε an (n, M, ε)*-code*. For any $\varepsilon \in [0, 1)$, a rate R is said to be ε*-achievable* if there exists a sequence of (n, M_n, ε_n)-codes such that $\liminf_{n\to\infty} \frac{1}{n}\log M_n \ge R$ and $\limsup_{n\to\infty} \varepsilon_n \le \varepsilon$. The ε*-capacity* for the AWGN EH channel, denoted by C_ε, is defined to be $C_\varepsilon \triangleq \sup\{R : R \text{ is } \varepsilon\text{-achievable}\}$. The following theorem is the main result in this paper. The proof sketch is provided in Section III.

Theorem 1: Let $\varepsilon \in (0, 1)$, and define

$$a \triangleq \max\left\{\mathrm{E}_{p_{E_1}}[E_1^2], 12\sqrt{2}P^2\right\}. \tag{5}$$

Suppose $n \ge 3$ is a sufficiently large integer such that

$$\frac{n}{\log n} \ge \max\left\{\mathrm{E}_{p_{E_1}}[E_1^2]/P^2, 12\sqrt{2}\right\}, \tag{6}$$

$$n \ge \big(\log(2+\varepsilon) - \log(\varepsilon^2)\big)^4 \tag{7}$$

and

$$n\log n \ge e^{0.4}(2+\varepsilon)/\varepsilon. \tag{8}$$

Then, there exists an $(n + m, M, \varepsilon)$-code such that

$$\log M \ge \frac{n}{2}\log(1+P) - \sqrt{\frac{(2+\varepsilon)nP}{\varepsilon(P+1)}} - n^{\frac{1}{4}} - 1 \tag{9}$$

where

$$m \triangleq \left\lceil 6\sqrt{an\log n}/P \right\rceil \tag{10}$$

denotes the length of the initial saving period before any transmission occurs and n denotes the length of the actual transmission period. In particular, there exists an (n^*, M, ε)-code with $n^* \triangleq n + m$ such that

$$\log M \ge \frac{n^*}{2}\log(1+P) - \frac{3\log(1+P)\sqrt{an^*\log n^*}}{P} - \sqrt{\frac{(2+\varepsilon)n^*P}{\varepsilon(P+1)}} - (n^*)^{\frac{1}{4}} - \frac{1}{2}\log(1+P) - 1. \tag{11}$$

Remark 1: Since $\frac{1}{n^*}\sum_{k=1}^{n^*} E_k$ converges to $\mathrm{E}_{p_{E_1}}[E_1] = P$ with probability one by the strong law of large numbers, it follows from the power constraint (3) and the strong converse theorem for the AWGN channel [6,7] that the ε-capacity of the AWGN EH channel is upper bounded by $\frac{1}{2}\log(1+P)$. Therefore, by normalizing both side of (11) by n and taking the limit, we see that Theorem 1 implies that the ε-capacity is

$$C_\varepsilon = C = \frac{1}{2}\log(1+P), \qquad \forall \varepsilon \in [0,1).$$

Remark 2: The investigation of the save-and-transmit scheme by Ozel and Ulukus in [1, Lemma 2] implies that $\frac{n^*}{2}\log(1+P) - O(\sqrt{n^*}(\log n^*)^\alpha)$ nats is achievable over n channel uses for any $\alpha > 1$ and for $n \to \infty$. Theorem 1 improves the lower bound of the second-order term because the backoff term improves from $-O(\sqrt{n^*}(\log n^*)^\alpha)$ to $-O(\sqrt{n^* \log n^*})$.

Remark 3: It follows from (5) and (11) that the coefficient of the second-order term achieved by the save-and-transmit strategy is at least

$$\nu \triangleq -\frac{3\log(1+P)}{P}\sqrt{\max\left\{\mathrm{E}_{p_{E_1}}[E_1^2], 12\sqrt{2}P^2\right\}}.$$

By inspecting the equations from (33) to (35) in the proof sketch of the theorem, we can see that (11) is a direct consequence of (9) and the second-order term in (11) is due to the saving period only, which means ν is affected by the length of the saving period m alone (but not ε). As P increases, the magnitude of ν increases and hence a longer saving period is required to guarantee a certain probability of *outage*, namely that the transmitted codeword does not satisfy all the EH constraints. This corroborates the fact that as P increases, the variance of each Gaussian codeword increases and hence a longer saving period is required to maintain a certain outage probability. Similarly, as $\mathrm{E}_{p_{E_1}}[E_1^2]$ increases while P is fixed, the variance of the energy arrival process is larger and hence a longer saving period is required to maintain a certain outage probability.

B. Numerical Results

In this section, we illustrate achievable rates as a function of n per Theorem 1. More specifically, we define

$$R_{n,\varepsilon}^{(\mathrm{EH})} \triangleq \frac{\frac{n}{2}\log(1+P) - \sqrt{\frac{(2+\varepsilon)nP}{\varepsilon(P+1)}} - n^{\frac{1}{4}} - 1}{n+m} \tag{12}$$

to be the non-asymptotic rate achievable by save-and-transmit according to (9) and (10). We plot $R_{n,\varepsilon}$ against n in Figure 1 for $\mathrm{E}[E_1] = P = 3\mathrm{dB}$ and various values of ε and $\mathrm{Var}[E_1] = \mathrm{E}[E_1^2] - P^2$, corresponding to the lines indicated as "(EH)" respectively. In order to demonstrate how much the EH constraints (3) degrade the non-asymptotic achievable rates compared to the peak power constraint

$$\Pr\left\{\sum_{k=1}^n X_k^2 \le nP\right\} = 1, \tag{13}$$

in Figure 1 we also plot the optimal transmission rate $R_{n,\varepsilon}^{(\mathrm{No\text{-}EH})}$ under the peak power constraint (13), corresponding to the lines indicated by "(No-EH)". Due to Polyanskiy-Poor-Verdú [3, Th. 54, Eq. (294)] and Tan-Tomamichel [8, Th. 1],

$$R_{n,\varepsilon}^{(\mathrm{No\text{-}EH})} = C + \sqrt{\frac{\mathrm{V}(P)}{n}}\Phi^{-1}(\varepsilon) + \frac{\log n}{2n} + O\left(\frac{1}{n}\right) \tag{14}$$

where $\mathrm{V}(P) \triangleq \frac{P(P+2)(\log e)^2}{2(P+1)^2}$ is known as the Gaussian *dispersion* function and Φ^{-1} is the inverse of the cumulative distribution function for the standard Gaussian distribution. We ignore the final correction term in (14) when we plot $R_{n,\varepsilon}^{(\mathrm{No\text{-}EH})}$ because it is negligible compared with the first three terms.

In Figure 1(a), we see that as ε increases, the backoff of both $R_{n,\varepsilon}^{(\mathrm{EH})}$ and $R_{n,\varepsilon}^{(\mathrm{No\text{-}EH})}$ from the capacity decreases, which is due to the increase of the magnitude of the second term in (12). In Figure 1(b), we see that as $\mathrm{Var}[E_1]$ increases for a fixed $\mathrm{E}[E_1]$, the backoff from the capacity increases, which is due to the explanation in Remark 3 that a longer saving period is required as $\mathrm{E}[E_1^2]$ increases. As we can see from Figures 1(a) and 1(b), the performance degradation due to the EH constraints and the save-and-transmit strategy compared to the peak power constraint is significant.

III. Save-and-Transmit Strategy

In this section, we investigate the save-and-transmit scheme proposed in [1, Sec. IV] in the finite blocklength regime and use this achievability scheme to prove Theorem 1. Due to space limitation, only a proof sketch of Theorem 1 is provided. The complete proof can be found in the long version of this paper [9]. The proof of Theorem 1 relies on the following result which is useful for obtaining a lower bound on the length of the energy-saving phase.

Lemma 1 ([9, Sec. III-A]): Suppose X^n and E^{m+n} are two random tuples, each consisting of i.i.d. random variables such that $X_1 \sim \mathcal{N}(x_1; 0, P)$, $\Pr_{p_{E_1}}\{E_1 < 0\} = 0$, and $\mathrm{E}_{p_{E_1}}[E_1] = \mathrm{E}_{p_{X_1}}[X_1^2] = P$. Define a as in (5). Then,

$$\Pr_{p_{X^n} p_{E^{m+n}}}\left\{\bigcup_{k=1}^n \left\{\sum_{\ell=1}^k X_\ell^2 \ge \sum_{\ell=1}^{m+k} E_\ell\right\}\right\} \le \left(\frac{e^{0.4}}{\log n}\right) e^{2\log n - \frac{mP}{2}\sqrt{\frac{\log n}{an}}}$$

for all sufficiently large n that satisfies (6).

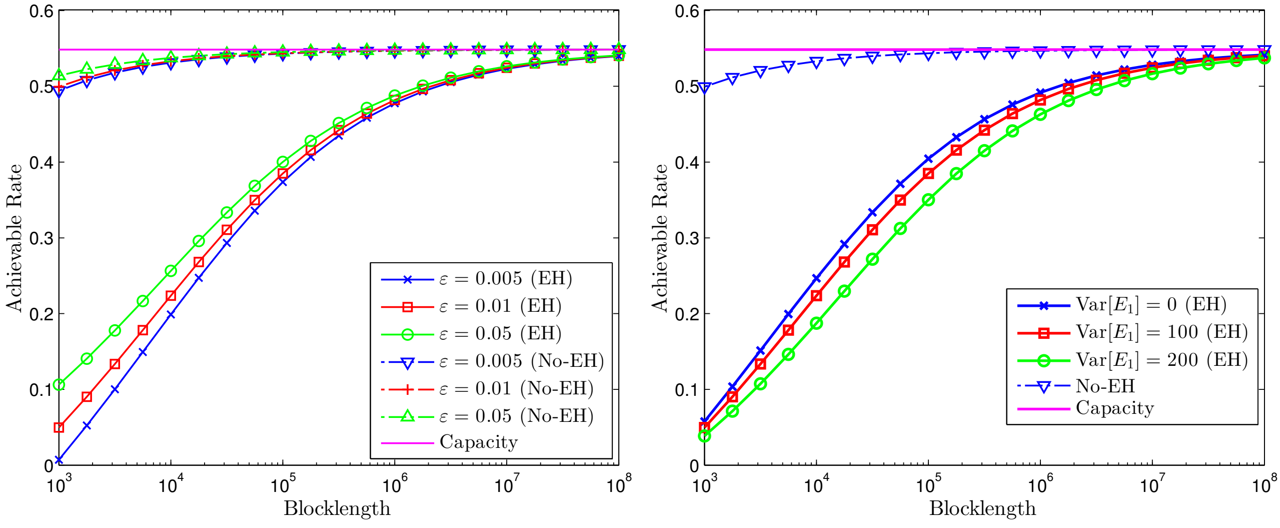
Proof sketch of Theorem 1: Fix an $\varepsilon \in (0,1)$. Define a as in (5). Fix a sufficiently large $n \ge 3$ such that (6), (7) and (8) hold. Define m as in (10) to be the number of time slots used for saving energy. Consider the random code that uses the channel $m+n$ times as follows:

Save-and-Transmit Random Codebook Construction

Let $\mathbf{0}^m$ denote the length-m zero tuple. Define

$$p_X(x) \triangleq \mathcal{N}(x; 0, P) \tag{15}$$

to be the distribution of a zero-mean Gaussian random variable X with variance P. In addition, let p_{X^n} be the product distribution of n independent copies of X. Construct M i.i.d. random tuples denoted by $X^n(1), X^n(2), \ldots, X^n(M)$ such that $X^n(1)$ is distributed according to p_{X^n}, where M will



(a) $\mathrm{E}[E_1] = P = 3\,\mathrm{dB}$ and $\mathrm{Var}[E_1] = 100$

(b) $\mathrm{E}[E_1] = P = 3\,\mathrm{dB}$ and $\varepsilon = 0.01$

Fig. 1. Achievable rates for the save-and-transmit scheme in (12) when the error probability is varied (left) and when the variance of the EH process is varied (right). On the plot on the right (Figure 1(b)) for the No-EH line, the peak power (cf. (13)) is kept at $P = 3\,\mathrm{dB}$

be carefully chosen later when we evaluate the probability of decoding error. Define

$$\tilde{X}^{m+n}(i) \triangleq (\mathbf{0}^m, X^n(i)) \tag{16}$$

for each $i \in \{1, 2, \ldots, M\}$ and construct the random codebook

$$\{\tilde{X}^{m+n}(i) \,|\, i \in \{1, 2, \ldots, M\}\}. \tag{17}$$

The codebook is revealed to both the encoder and the decoder. To facilitate discussion, we let $X_k(i)$ and $\tilde{X}_k(i)$ denote the k^{th} symbols in $X^n(i)$ and $\tilde{X}^{m+n}(i)$ respectively for each i. Since the first m symbols of each random codeword $\tilde{X}^{m+n}(i)$ are zeros by (16), the source will just transmit 0 with probability one until time slot $m+1$ when the amount of energy $\sum_{k=1}^{m+1} E_k$ is available for encoding $\tilde{X}_{m+1}(W) \stackrel{(16)}{=} X_1(W)$.

Encoding under the EH Constraints

The source s has the knowledge of E^k before transmitting its symbol in time slot k for each $k \in \{1, 2, \ldots, m+n\}$. For each $i \in \{1, 2, \ldots, M\}$, recalling that $\tilde{X}_k(i)$ is the k^{th} element of $\tilde{X}^{m+n}(i) \stackrel{(16)}{=} (\mathbf{0}^m, X^n(i))$, we construct recursively for $k = 1, 2, \ldots, m+n$ the random variable

$$\hat{X}_k(i, E^k) \triangleq \begin{cases} \tilde{X}_k(i) & \text{if } (\tilde{X}_k(i))^2 \le \sum\limits_{\ell=1}^{k} E_\ell - \sum\limits_{\ell=1}^{k-1} (\hat{X}_\ell(i, E^\ell))^2, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

To send message W which is uniformly distributed on $\{1, 2, \ldots, M\}$, node s transmits $\hat{X}_k(W, E^k)$ in time slot k for each $k \in \{1, 2, \ldots, m+n\}$. Since node s transmits 0 with probability one in the first m times slots by (16) and (18), the transmitted codeword $(\hat{X}_1(W, E^1), \hat{X}_2(W, E^2), \ldots, \hat{X}_{m+n}(W, E^{m+n}))$ satisfies the EH constraints (3) by (18).

Threshold Decoding

Upon receiving

$$\hat{Y}^{m+n} = \hat{X}^{m+n}(W, E^{m+n}) + Z^{m+n} \tag{19}$$

where

$$\hat{X}^{m+n}(W, E^{m+n}) \triangleq (\hat{X}_1(W, E^1), \hat{X}_2(W, E^2), \ldots, \hat{X}_{m+n}(W, E^{m+n})) \tag{20}$$

denotes the transmitted tuple specified in (18) and $Z^{m+n} \sim \mathcal{N}(z^{m+n}; 0, 1)$ by the channel law (cf. (4)), node d constructs its subtuple denoted by $\bar{Y}^n$ by keeping only the last n symbols of $\hat{Y}^{m+n}$. Recalling that $q_{Y|X}$ denotes the channel law and p_X was defined in (15), we define the joint distribution

$$p_{X,Y} \triangleq p_X q_{Y|X}, \tag{21}$$

and define p_{X^n,Y^n} as

$$p_{X^n,Y^n}(x^n, y^n) \triangleq \prod_{k=1}^{n} p_{X,Y}(x_k, y_k)$$

for all $(x^n, y^n) \in \mathbb{R}^2$. Then, node d declares $\varphi(\bar{Y}^n) \in \{1, 2, \ldots, M\}$ (with a slight abuse of notation, we write $\varphi(\bar{Y}^n)$ instead of $\varphi(\hat{Y}^{m+n})$) to be the transmitted message where $\varphi(\bar{Y}^n)$ is the decoding function defined as follows: If there exists a unique index j such that

$$\log\left(\frac{p_{Y^n|X^n}(\bar{Y}^n|X^n(j))}{p_{Y^n}(\bar{Y}^n)}\right) > \log M + n^{\frac{1}{4}}, \tag{22}$$

then $\varphi(\bar{Y}^n)$ is assigned the value j. Otherwise, $\varphi(\bar{Y}^n)$ is assigned a random value uniformly distributed on $\{1, 2, \ldots, M\}$.

Probability of Violating the EH Constraints

Defining $\bar{X}^n(W, E^{m+n})$ to be the tuple containing the last n symbols of $\hat{X}^{m+n}(W, E^{m+n})$, we obtain from (20), (18) and (16) that

$$\Pr\left\{\begin{matrix}\bar{X}^n(W,E^{m+n})\\ = X^n(W)\end{matrix}\middle|\bigcap_{k=1}^{n}\left\{\sum_{\ell=1}^{k}(X_\ell(W))^2 \le \sum_{\ell=1}^{m+k}E_\ell\right\}\right\} = 1. \tag{23}$$

Combining Lemma 1, (5) and (6) and noting that E^{m+n} and $(W, X^n(W))$ are independent by construction, we obtain

$$\Pr\left\{\bigcap_{k=1}^{n}\left\{\sum_{\ell=1}^{k}(X_\ell(W))^2 \le \sum_{\ell=1}^{m+k}E_\ell\right\}\right\} \ge 1-\left(\frac{e^{0.4}}{\log n}\right)e^{2\log n-\frac{mP}{2}\sqrt{\frac{\log n}{an}}}. \tag{24}$$

Combining (23) and (24) and using the bound on m in (10) and the bound on $n\log n$ in (8), we can obtain

$$\Pr\left\{\bar{X}^n(W,E^{m+n}) = X^n(W)\right\} \ge 1-\frac{\varepsilon}{2+\varepsilon}. \tag{25}$$

Calculating the Probability of Decoding Error

Defining $\bar{Z}^n$ to be the tuple containing the last n symbols of Z^{m+n} and recalling $\bar{X}^n(W,E^{m+n})$ and $\bar{Y}^n$ are the tuples containing the last n symbols of $\hat{X}^{m+n}(W,E^{m+n})$ and $\hat{Y}^{m+n}$ respectively, we obtain from (19) and (25) that

$$\Pr\left\{\bar{Y}^n = X^n(W)+\bar{Z}^n\right\} \ge 1-\frac{\varepsilon}{2+\varepsilon}, \tag{26}$$

where $X^n(W)$ and $\bar{Z}^n$ are independent. Let $\mathcal{E}_{i|w}$ denote

$$\left\{\log\left(\frac{p_{Y^n|X^n}(X^n(w)+\bar{Z}^n|X^n(i))}{p_{Y^n}(X^n(w)+\bar{Z}^n)}\right) \le \log M + n^{\frac{1}{4}}\right\}. \tag{27}$$

Using the Shannon's bound [10] and (7) and recalling the symmetry of the random codebook, after some calculations we can obtain for each w

$$\Pr_{p_W(\prod_{i=1}^{M}p_{X^n(i)})p_{\bar{Z}^n}}\left\{\mathcal{E}_{w|w}\cup\bigcup_{j\in\{1,2,\ldots,M\}\setminus\{w\}}\mathcal{E}_{j|w}^c\middle|W=w\right\} \le \Pr_{\prod_{k=1}^{n}p_{X_k(1)}p_{\bar{Z}_k}}\left\{\mathcal{E}_{1|1}\right\}+\frac{\varepsilon^2}{2+\varepsilon}. \tag{28}$$

In order to obtain a simple upper bound on the first term in (28), we choose M to be the unique integer that satisfies

$$\begin{aligned}&\log(M+1)\\ &\ge n\mathrm{E}_{p_{X,Y}}\left[\log\left(\frac{p_{Y|X}(Y|X)}{p_Y(Y)}\right)\right]\\ &\quad-\sqrt{\frac{(2+\varepsilon)n}{\varepsilon}\mathrm{Var}_{p_{X,Y}}\left[\log\left(\frac{p_{Y|X}(Y|X)}{p_Y(Y)}\right)\right]}-n^{\frac{1}{4}}\\ &>\log M.\end{aligned} \tag{29}$$

Using (27), (29) and the Chebyshev's inequality, we have

$$\Pr_{\prod_{k=1}^{n}p_{X_k(1)}p_{\bar{Z}_k}}\left\{\mathcal{E}_{1|1}\right\} \le \frac{\varepsilon}{2+\varepsilon}. \tag{30}$$

The decoding error probability can now be bounded above as

$$\begin{aligned}&\Pr_{p_{W,X^n(W)}p_{\bar{Z}^n}p_{\bar{Y}^n|W,X^n(W),\bar{Z}^n}}\left\{\varphi(\bar{Y}^n)\ne W\right\}\\ &\overset{(26)}{\le}\Pr\left\{\{\varphi(\bar{Y}^n)\ne W\}\cap\{\bar{Y}^n = X^n(W)+\bar{Z}^n\}\right\}+\frac{\varepsilon}{2+\varepsilon}\\ &\le \Pr\left\{\varphi(X^n(W)+\bar{Z}^n)\ne W\right\}+\frac{\varepsilon}{2+\varepsilon} \qquad (31)\\ &\overset{(a)}{\le}\varepsilon \qquad (32)\end{aligned}$$

where (a) follows from the threshold decoding rule (cf. (22) and (27)), (28) and (30). Using (10), (17), (29) and (32), we conclude that the constructed code is an $(n+m, M, \varepsilon)$-code that satisfies (29), which implies from (21), (15) and (4) that

$$\log M + 1 \ge \frac{n}{2}\log(1+P)-\sqrt{\frac{(2+\varepsilon)nP}{\varepsilon(P+1)}}-n^{\frac{1}{4}} \tag{33}$$

$$\ge \frac{n^*-m}{2}\log(1+P)-\sqrt{\frac{(2+\varepsilon)n^*P}{\varepsilon(P+1)}}-(n^*)^{\frac{1}{4}} \tag{34}$$

where $n^* \triangleq n+m$. Equation (9) then follows from (33). Since $m\ge 0$ and $m \le \frac{6\sqrt{an\log n}}{P}+1$ by (10), it follows that

$$\frac{6\sqrt{an^*\log n^*}}{P}+1 \ge \frac{6\sqrt{an\log n}}{P}+1 \ge m. \tag{35}$$

Equation (11) then follows from (34) and (35). ■

Acknowledgements

Silas Fong and Vincent Tan gratefully acknowledge financial support from the National University of Singapore (NUS) under grant R-263-000-A98-750/133 and NUS Young Investigator Award R-263-000-B37-133. Jing Yang is supported by United States National Science Foundation (NSF) under grants ECCS-1405403 and ECCS-1454471.

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