EE 351

MIDTERM #2

SPRING 2004

Name: KEY

NOTE:
1. Exam is comprised of 5 problems, each with multiple parts.
2. Point values are given in parentheses for each part.
3. You are allowed one 8 1/2 X 11 sheet of paper and a calculator
4. You must show all work and write your answers in the spaces provided.
5. You have 2 hours to complete the exam.

DO NOT WRITE BELOW THIS LINE:

Problem 1 (15 points) : _____
Problem 2 (20 points) : _____
Problem 3 (25 points) : _____
Problem 4 (30 points) : _____
Problem 5 (10 points) : _____

TOTAL (100 points) : _____

Mean: 82
High: 96
Best improvement: 23 (from 0)
Biggest drop: -25 (from 20)
1. 
   a) Determine whether the discrete-time system below is linear/non-linear, time-invariant/time-varying, causal/non-causal and BIBO stable/unstable. (8 points)

   \[
   y[n] = \begin{cases} 
   -10, & x[n-1] < -2 \\
   (n+1) \cdot x[n-1], & -2 \leq x[n-1] \leq 2 \\
   10, & x[n-1] > 2 
   \end{cases} 
   \]

   ANSWER (circle one in each group): linear / non-linear  
   time-invariant/ time-varying  
   causal/ non-causal  
   BIBO stable/ BIBO unstable

   Due to (n+1) term
   Due to saturation at ±10 for |x| > 2
   y[n] depends only on x[n-1]

   b) Determine whether the discrete-time LTI system described by the following impulse response is causal/non-causal and BIBO stable/unstable. (4 points)

   \[
   h[n] = (5)^n \cdot u[3-n] 
   \]

   ANSWER (circle one in each group): causal / non-causal  
   BIBO stable/ BIBO unstable

   Exists for n ≤ 3. Since it exists for n < 0, system is non-causal.
   \[
   \begin{align*}
   \left( \frac{3}{5} \right)^{5} & = \frac{5^5}{5^5} \\
   & = 156.25
   \end{align*}
   \]

   c) Consider the following 2 statements:
   I) LTI systems described by non-recursive difference equations are always stable
   II) LTI systems described by non-recursive difference equations are always causal

   Which statement(s) is/are true? (3 points)

   ANSWER (circle one): I only  II only  I and II  neither I nor II
2. In honor of the upcoming birth of my first child, I’ve decided to have a “baby-related” question this year.

The weight (in ounces) of liquid “output” produced by a newborn on day \( n \) is given by \( p[n] \) (note the appropriate use of variable name!). After extensive diaper changes, it is determined that the amount of output produced by a particular infant in a given day is equal to 60% of the amount of input \( m[n] \) (milk) ingested that day by the infant plus 20% of the amount of milk ingested the prior day (because it takes some of the milk longer to go through the system), minus 10% of the amount of “output” produced the prior day (because a baby is less likely to produce as much output if he/she produced a large amount the day before).

a) Based on the information above, determine a difference equation relating output \( p[n] \) to input \( m[n] \). (4 points)

\[
\text{ANSWER: } p[n] = -0.1 p[n-1] + 0.6 m[n] + 0.2 m[n-1]
\]

b) Assume that the output on day -1 (the day before feeding begins) is \( p[-1] = 10 \) (ounces). Assume that the amount of milk (in ounces) consumed the first few days of the baby’s life is given by

\[
m[n] = [16 \ 18 \ 20 \ 22 \ 22 \ 22 \ldots];
\]

Determine, iteratively, the number of ounces of output produced in days 0-2. (6 points)

\[
\begin{aligned}
  n=0 & \quad p[0] = -0.1(10) + 0.6(16) = 8.6 \\
  n=1 & \quad p[1] = -0.1(8.6) + 0.6(18) + 0.2(16) = 13.14 \\
  n=2 & \quad p[2] = -0.1(13.14) + 0.6(20) + 0.2(18) = 14.286 \\
\end{aligned}
\]

\[
\text{ANSWER: } p[0] = 8.6 \quad \quad p[1] = 13.14 \quad \quad p[2] = 14.286
\]
c) Suppose that the input/output relationship for a *different* infant is given by

\[ p[n] = 0.25 p[n-1] + 0.5m[n]. \]

If \( p[-1] = 8 \) and \( m[n]=20\cdot u[n] \), determine a closed-form expression (equation) for \( p[n] \) that is valid for \( n \geq 0 \). You may use any method you want to find \( p[n] \), but you must show/explain your work. (8 points)

\[ \text{Char Eqn: } E - 0.25 = 0 \quad y = 0.25 \quad \text{so } \gamma^n = C(0.25)^n \]

\[ \begin{align*}
\text{Using IC, } y &= C(0.25)^n \rightarrow C = 2 \rightarrow P_{m=0}[n] = 2(0.25)^n \cdot u[n] \\
\end{align*} \]

\[ h[n] = C(0.25)^n \]

\[ h[0] = 0.5 \cdot 0.25 = 0.125 \rightarrow 0.5 = C(0.25)^0 \rightarrow C = 5 \rightarrow h[n] = 0.5(0.25)^n \cdot u[n] \]

\[ \begin{align*}
P_{l=0} &= h[n] + m[n] = \sum_{k=0}^{n} 5(0.25)^k &= 10 \cdot \frac{(0.25)^0 - (0.25)^{n+1}}{1 - 0.25} \\
&= \frac{40}{3} \cdot (1 - (0.25)^n) \cdot u[n] \\
\end{align*} \]

**ANSWER:** \( p[n] = \left[ \frac{40}{3} - \frac{4}{3}(0.25)^n \right] \cdot u[n] \)

d) If the amount of ingested milk in problem c) was \( m[n]=10\cdot u[n] \) instead of \( m[n]=20\cdot u[n] \), would your equation for \( p[n] \) in part c) be halved as well? Why or why not? (2 points)

**ANSWER:** No. \( P_{l=0} \) is linear w.r.t. \( m[n] \), but \( P_{m=0} \) is linear w.r.t. IC so just halving \( m[n] \) will not halve \( \sum_{n=0}^{\text{TOTAL}} p[n] \)

e) What should I name my baby-to-be? It is going to be a boy. TBD
3. EACH PART BELOW IS INDEPENDENT UNLESS STATED OTHERWISE

a) Analytically determine an expression for the zero-input response of the following difference equation. Show and explain all work. (7 points)

\[ y[n] = \frac{1}{2} y[n-1] - \frac{1}{16} y[n-2] + 2x[n], \quad \text{initial conditions: } y[-2] = 16, \quad y[-1] = 3 \]

\[
\text{Char Eqn:} \quad E^2 - \frac{1}{2} E + \frac{1}{16} = 0 \Rightarrow (E - \frac{1}{4})(E - \frac{1}{4}) = 0 \quad \Rightarrow E = \frac{1}{4}, \frac{1}{4}
\]

\[ y[n] = \left[ C_1 \left(\frac{1}{4}\right)^n + C_2 n \left(\frac{1}{4}\right)^n \right] u[n] \]

Plug in ICs.

\[ n = -1: \quad 3 = C_1 \left(\frac{1}{4}\right) + C_2 (-1) \left(\frac{1}{4}\right) \Rightarrow 4C_1 - 4C_2 = 3 \rightarrow C_1 = \frac{1}{2}, C_2 = \frac{1}{4} \]

\[ n = -2: \quad 16 = C_1 (16) + C_2 (-2) (16) \quad 16C_1 - 32C_2 = 16 \rightarrow C_2 = \frac{1}{4} \]

\[ \text{ANSWER: } y_x = \left[ \frac{1}{2} \left(\frac{1}{4}\right)^n - \frac{1}{4} n \left(\frac{1}{4}\right)^n \right] u[n] \]

b) If a discrete-time LTI system is given by the difference equation:

\[ y[n] = y[n-2] + x[n-2] \]

\[ N = 2 \quad M = 2 \]

determine the impulse response of this system. Your answer must be a closed-form equation. Show and explain all work. (9 points)

\[ \text{Char Eqn:} \quad E^2 - 1 = 0 \Rightarrow (E + 1)(E - 1) = 0 \]

so

\[ h[n] = \left[ C_1 (-1)^n + C_2 (1)^n + A_0 \delta[n] \right] u[n] \]

Need 3 values for \( h[n] \)

\[ h[0] = h[-2] + \delta[-2] = 0 \]

\[ 0 = C_1 + C_2 + A_0 \quad C_1 = \frac{1}{2} \]

\[ h[1] = h[-1] + \delta[-1] = 0 \]

\[ 0 = -C_1 + C_2 \quad C_2 = \frac{1}{2} \]

\[ h[2] = h[0] + \delta[0] = 1 \]

\[ 1 = C_1 + C_2 \quad A_0 = -1 \]

\[ \text{ANSWER: } h[n] = \left[ \frac{1}{2} (-1)^n + \frac{1}{2} - 1 \delta[n] \right] u[n] \]
c) An LTI discrete-time system is described by the difference equation
\[ y[n] + a \cdot y[n-1] + b \cdot y[n-2] = x[n] \] with initial conditions \( y[-1] = -4 \) and \( y[-2] = 8 \).

Determine values for \( a \) and \( b \) if the zero-input response for \( n \geq 0 \) is given by
\[ y_{x=0}[n] = 4 \cdot \left( \frac{1}{2} \right)^n \cdot \cos\left( \frac{n}{3} \pi - \pi \right) \]
(6 points)

\[
\text{roots of char eqn: } \left( E - \frac{1}{2} e^{j \frac{\pi}{3}} \right) \left( E - \frac{1}{2} e^{-j \frac{\pi}{3}} \right) = E^2 - \frac{1}{2} \cdot 2 \cos \frac{\pi}{3} E + \frac{1}{4}
\]
\[
= E^2 - \frac{1}{2} E + \frac{1}{4}
\]
\[ a = \frac{1}{2}, \quad b = \frac{1}{4} \]

ANSWER: \( a = \frac{1}{2}, \quad b = \frac{1}{4} \)

d) In part c), what part(s) of \( y_{x=0}[n] \) would change if the initial conditions were different? (3 points)

The amplitude \( 4 \) and phase angle \( -\pi \)
would change if \( IC \) were different.

ANSWER: ________________________________

______________________________

______________________________
4. EACH PART BELOW IS INDEPENDENT.

a) Consider the discrete-time LTI system with input $x[n]$ and system impulse response $h[n]$ given by:

$$x[n] = [0, 1, 2, 3] \quad h[n] = [2, 1, -1]$$

Using graphical (sliding bar) convolution, determine the zero-state output $y_{ic=0}[n] = x[n] * h[n]$. (8 points)

\[\longleftrightarrow -1 \quad 1 \quad 2 \quad 3\]

ANSWER: $y_{ic=0}[n] = [2, 5, 7, 1, -3]$.

b) If the input signal, $x[n]$, and system impulse response, $h[n]$, are given as shown below, use analytical convolution to determine an equation for the zero-state output signal $y_{ic=0}[n]$ that is valid for all $n$. You must show and explain all work to receive full credit. (12 points)

$$x[n] = \left(-\frac{3}{4}\right)^n \cdot u(n + 2) \quad h[n] = \left(\frac{3}{2}\right)^n \cdot u(-n + 1)$$

$$y[n] = \sum_{k=-\infty}^{\infty} \left(-\frac{3}{4}\right)^k \cdot u(k+2) \cdot \left(\frac{3}{2}\right)^{n-k} \cdot u(-k+1+n)$$

Case 1: $n-1 \geq -2 \Rightarrow n \geq -1$

$$\left(\frac{3}{2}\right)^n \sum_{k=-\infty}^{\infty} \left(-\frac{1}{2}\right)^k = \left(\frac{3}{2}\right)^n \frac{\left(-\frac{1}{2}\right)^{-1} - 1}{1 + \frac{1}{2}} = \frac{2}{3} \left(\frac{3}{2}\right)^n \left(-2\right) \left(-\frac{1}{2}\right)^n = -\frac{2}{3} \left(\frac{3}{2}\right)^n$$

Case 2: $n-1 < -2 \Rightarrow n < -1$

$$\left(\frac{3}{2}\right)^n \sum_{k=-2}^{\infty} \left(-\frac{1}{2}\right)^k = \frac{\left(\frac{3}{2}\right)^n \left(-\frac{1}{2}\right)^2}{1 + \frac{1}{2}} = \frac{8}{3} \left(\frac{3}{2}\right)^n$$

ANSWER: $y_{ic=0}[n] = \begin{cases} 
\frac{8}{3} \left(\frac{3}{2}\right)^n, & n < -1 \text{ or } n \leq -1 \\
-\frac{4}{3} \left(\frac{3}{4}\right)^n, & n \geq -1 \text{ or } n \geq 0
\end{cases}$
c) Consider the parallel interconnection of two causal discrete-time LTI sub-systems as shown in the figure below:

\[
\begin{array}{c}
 x[n] \\
 h_1[n] \\
 h_2[n] \\
 y[n]
\end{array}
\]

If \( h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2] \) and the overall impulse response is \( h[n] = [1 \ 5 \ 7 \ 3] \), determine the impulse response of the first sub-system, \( h_1[n] \). (4 points)

\[
 h[n] = h_1[n] + h_2[n] \Rightarrow h_1[n] = h[n] - h_2[n] = \begin{bmatrix} 1 & 5 & 7 \ 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \ 1 \end{bmatrix}
\]

Answer: \( h_1[n] = \begin{bmatrix} 0 & 3 & 6 \ 3 \end{bmatrix} \)


d) Repeat c) if \( h_1[n] \) and \( h_2[n] \) are connected in cascade instead of parallel. (6 points)

\[
 h[n] = h_1[n] + h_2[n]
\]

Since \( \text{length}[h] = \text{length}(h_1) + \text{length}(h_2) - 1 \Rightarrow \text{length}(h_1) = 2 \)

Furthermore, \( h_1[n] \) must start at \( n = 0 \)

\[
 h_1 = [a \ b]
\]

Using sliding bar convolutions: \( a(1) = 1 \Rightarrow a = 1 \)

\( a(2) + b(1) = 5 \Rightarrow b = 3 \)

\( \Rightarrow h_1[n] = \begin{bmatrix} 1 & 3 \end{bmatrix} \) (can check other values for \( h[n] \) to make sure)

Answer: \( h_1[n] = \begin{bmatrix} 1 & 3 \end{bmatrix} \)
5. Consider the discrete-time sequence \( x[n] = [1 \ 2 \ 2 \ 1] \)

\[ D_1 = \frac{1}{4} \sum_{\gamma=1}^{\infty} x(\gamma) e^{-j\frac{\pi}{4} \gamma} = \frac{1}{4} \left[ 1 + 2e^{-j\frac{\pi}{4}} + 2e^{-j\frac{\pi}{4}} + 1e^{-j\frac{3\pi}{4}} \right] = \frac{1}{4} \left[ 1 - 2 + 2 + 1 \right] = \frac{1}{4} [-1 - j] \]

**ANSWER:** \( D_1 = \frac{-1 - j}{4} \).

b) If the sequence above repeated with a period of \( N_0 = 8 \), how many harmonics (including DC) are in the Fourier Series expansion? (select 1) (2 points)

i) There are always an infinite number of harmonics in a periodic signal

ii) The number of harmonics is always equal to the number of terms in the original sequence, so the number of harmonics = 4.

iii) The number of harmonics is always equal to the period of the sequence, so the number of harmonics = 8.

iv) The number of harmonics is always equal to \( N_0 - 1 \) because the DTFS summation goes from \( n=0 \) to \( N_0-1 \), so the number of harmonics = 7.

c) If the sequence \( x[n] \) is NOT periodic, determine an expression for the discrete-time Fourier transform \( X(\Omega) \). (4 points)

\[ X(\Omega) = \sum_{n=0}^{3} x(\gamma) e^{-j\Omega n} = 1 + 2e^{-j\Omega} + 2e^{-j2\Omega} + e^{-j3\Omega} \]

**ANSWER:** \( X(\Omega) = 1 + 2e^{-j\Omega} + 2e^{-j2\Omega} + e^{-j3\Omega} \).