EE 351

Sample Exam Solutions

From FA98
1. Complex numbers (10 points)

Consider the complex function \( H(\theta) = \frac{S e^{j\theta}}{1 + e^{-j2\theta}} \)

All parts below are independent of each other.

a) Evaluate this function at \( \theta = \frac{\pi}{4} \). Show all work and express your answer in rectangular form.

(6 points)

\[
H\left(\frac{\pi}{4}\right) = \frac{5e^{j\pi/4}}{1 + e^{-j\pi/2}}
\]

But \( e^{-j\pi/2} = -j \)

So \( H\left(\frac{\pi}{4}\right) = \frac{5e^{j\pi/4}}{1 - j} = \frac{5e^{j\pi/4}}{\sqrt{2}e^{-j\pi/4}} = \frac{5}{\sqrt{2}} e^{j\pi/2} \)

But \( e^{j\pi/2} = j \)

So \( H\left(\frac{\pi}{4}\right) = \frac{5}{\sqrt{2}} j = 3.5355j \)

ANSWER: \( H\left(\frac{\pi}{4}\right) = \frac{5}{\sqrt{2}} j \)

b) Determine a value for \( \theta \) such that this function would evaluate to \( \infty \). (4 points)

\[
\text{Function } = \infty \text{ if denominator } = 0
\]

\( e^{-j2\theta} = -1 \)

But \( e^{j\pi} = -1 \) so \( -2\theta = \pm \pi \)

So \( \theta = \pm \pi/2 \)

ANSWER: \( \theta = \pm \pi/2 \)
2. Sampling and Quantization (20 points)

Consider the simplified A/D system given below, which performs sampling followed by quantization and encoding:

![Diagram of A/D system]

The quantization map \( Q(x) \) is given by:

![Diagram of quantization map]

*Note: If a data point falls on a quantization threshold, use the convention of selecting the larger level.*

All parts below are independent of each other.

a) If \( x_c(t) = 8 \sin(2\pi \cdot 500t - \pi/4) \) and the sampling period is \( T=0.0002 \) seconds/sample, determine the values of \( x_1[n] \) and \( x_2[n] \) for \( n=0, 1, 2, \) and \( 3 \). (8 points)

\[
X_1[n] = X_c(t) \bigg|_{t=nT} = 8 \sin \left(2\pi \cdot 500 \cdot n \cdot 0.0002 - \frac{\pi}{4}\right)
\]

\[
X_1[0] = 8 \sin \left(2\pi \cdot 0.1 \cdot 0 - \frac{\pi}{4}\right) \Rightarrow X_1[0] = -5.6569 \Rightarrow X_2[0] = -6
\]

\[
X_1[1] = 8 \sin \left(2\pi \cdot 0.1 \cdot 1 - \frac{\pi}{4}\right) \Rightarrow X_1[1] = -1.2515 \Rightarrow X_2[1] = -2
\]

\[
X_1[2] = 8 \sin \left(2\pi \cdot 0.1 \cdot 2 - \frac{\pi}{4}\right) \Rightarrow X_1[2] = 3.6319 \Rightarrow X_2[2] = 4
\]

\[
X_1[3] = 8 \sin \left(2\pi \cdot 0.1 \cdot 3 - \frac{\pi}{4}\right) \Rightarrow X_1[3] = 7.1281 \Rightarrow X_2[3] = 8
\]

**ANSWER:**

\[
\]

\[
x_2[0] = -6, x_2[1] = -2, x_2[2] = 4, x_2[3] = 8
\]
b) Determine a different input signal, \(x_{c2}(t)\), that would yield the same sample values that \(x_c(t)\) had in part (a). Assume that the sampling period is unchanged (\(T = 0.0002\) seconds/period). Your answer for \(x_{c2}(t)\) must have a different frequency than \(x_c(t)\) in part (a). (6 points)

Hint: You do not need the answer in (a) to solve this problem.

\[
\text{Since } 0 \text{ is indistinguishable from } 0+2\pi k, \text{ where } k \text{ is any integer, there are infinite number of solutions to this problem, one of them would be:}
\]

\[
8 \cdot \sin \left( \left(2\pi \cdot 0.1 + 2\pi \right) n - \frac{\pi}{4} \right) = 8 \cdot \sin \left( 2\pi 0.1n - \frac{\pi}{4} \right) \Rightarrow
\]

Convert back to C.T \( w = \frac{0}{T} \Rightarrow w = \frac{2\pi \cdot 0.1 + 2\pi}{0.0002} = 2\pi \cdot 5500 \text{ rad/sec} \)

ANSWER: \(x_{c2}(t) = 8 \cdot \sin \left( 2\pi \cdot 5500 t - \frac{\pi}{4} \right) \)

More general answer would be \(8 \cdot \sin \left( \left(2\pi \cdot 5500 + 2\pi k \right) t - \frac{\pi}{4} \right) \)


c) If some unknown input signal with maximum absolute value of 8 is applied to the quantizer, determine the maximum possible quantization error magnitude. (3 points)

Max. error occurs when an input data falls on a quantization threshold, for example \(4 \rightarrow 6 \Rightarrow \)

ANSWER: maximum error = \(2 \) \( \text{Error} = 6 - 4 = 2 \)

d) The purpose of the encoder is to convert each quantization level to an N-bit binary code. Determine the minimum number of encoder bits \(N\) so that each quantization level would be expressed by a unique binary code. (3 points)

Number of quantization levels = \(2^N\)

Here, there are 8 quantization levels \(\Rightarrow\)

\(8 = 2^3 \Rightarrow \text{Answer: } N = 3\)
3. Discrete-time signals and signal transformations (25 points)

All parts below are independent of each other.

a) Consider the discrete-time sequence given by:

\[ x[n] = [3, -4, 6, 2, -1, 5, -7, 8] \]

If \( y[n] = x[an+b] = [0, 3, 2, -7] \), determine the correct values for \( a \) and \( b \). (5 points)

Here \( x[n] \) shifted right by 5, then scaled by 3

\[ y[n] = x[3n - 5] \]

ANSWER: \( a = \frac{3}{5}, \ b = -5 \)

b) Consider 2 discrete-time signals \( x_1[n] \) and \( x_2[n] \). Show that if \( x_1[n] \) is an odd signal and \( x_2[n] \) is an even signal, then \( y[n] = x_1[n] \cdot x_2[n] \) is an odd signal. (3 points)

Since \( x_1[n] \) is odd we know that \( x_1[-n] = -x_1[n] \)

and \( " \) \( x_2[n] \) is even \( " \) \( x_2[-n] = x_2[n] \)

\[ y[-n] = x_1[-n] \cdot x_2[-n] = -x_1[n] \cdot x_2[n] = -y[n] \iff y[n] \text{ is an odd signal} \]

c) Determine and sketch the odd part of the discrete-time signal \( x[n] = [-3, 0, 1, 2, -2] \). (6 points)

\[
\begin{array}{cccccc}
x[n] & \rightarrow & -3 & 0 & 1 & 2 & -2 \\
x[-n] & \rightarrow & -2 & 2 & 1 & 0 & -3 \\
x[n]+x[-n] & \rightarrow & 2 & -2 & -4 & 0 & 4 & 2 & -2 \\
\Rightarrow x_{odd}[n] & = & \frac{x[n]+x[-n]}{2} \\
\Rightarrow x_{odd}[n] & = & [1, -1, -2, 0, 2, 1, -1]
\end{array}
\]
d) If \( \{x[n]\} = [5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5] \),

determine the discrete-time sequence given by \( x[n] \cdot \left\{ \sum_{k=2}^{\infty} \delta[n+3-k] - u[n-2] \right\} \).

Hint: Drawing figures may help. (6 points)

\[
\sum_{k=2}^{\infty} \delta[n+3-k] = u[n-2] - 1 = 1
\]

\Rightarrow x[n] \cdot \left\{ \sum_{k=2}^{\infty} \delta[n+3-k] - u[n-2] \right\} = x[n] \cdot \left\{ \delta[n+1], \delta[n], \delta[n-1] \right\} = x[n] \cdot \delta[n+1], x[0] \delta[n]

\text{ANSWER: } [1, 0, -1]

e) Determine whether each discrete-time sequence below is periodic or not. If it is, then determine the fundamental period \( N \).

i) \( x[n] = \cos(1.25\pi n + \frac{\pi}{4}) \) (3 points)

\[
\theta = 1.25\pi \Rightarrow \frac{\theta}{2\pi} = \frac{1.25\pi}{2\pi} = \frac{5}{8} = \frac{k}{N} \Rightarrow N = 8
\]

\text{ANSWER: fundamental period} = 8

ii) \( x[n] = 1 + e^{jn} \cos\left(\frac{n}{4}\right) \) (2 points)

\[
\theta = 2 \Rightarrow \frac{\theta}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi} \neq \frac{k}{N} = \text{integer} \Rightarrow
\]

\text{ANSWER: fundamental period} = \underline{\text{NOT PERIODIC!}} \quad \underline{\text{NOT PERIODIC!}}

\underline{\text{NOT PERIODIC!}}

\text{Not periodic because all terms must be periodic for the summation to be periodic!!}
4. Sampling, reconstruction and aliasing (25 points)

Consider a continuous-time signal \( x_c(t) = 4 \cos(2\pi \cdot 800t) - 4 \cos(2\pi \cdot 600t) \) passed through the ideal sampling/reconstruction system below:

\[
\begin{array}{c}
\text{Ideal Sampler} \\
\text{Sampling freq.} = F_s \\
\end{array} \quad \begin{array}{c}
\text{Ideal Lowpass} \\
\text{Reconstruction Filter} \\
\text{with cutoff frequency} \\
F_s/2 \text{ and } \text{Gain} = 1 \\
\end{array} \quad \rightarrow \quad x_r(t)
\]

All parts below are independent of each other.

a) If \( F_s = 1000 \text{ Hz} \), sketch \( X_c(j\omega) \) and \( X_r(j\omega) \) on the graphs below. From the graph of \( X_r(j\omega) \), determine the corresponding time-domain signal \( x_r(t) \). Make sure to label all key amplitudes. (10 points)

\[
x_r(t) = \frac{4 \cos 2\pi \cdot 200t}{2\pi \cdot 200} - \frac{4 \cos 2\pi \cdot 400t}{2\pi \cdot 400}
\]
b) Determine a different sampling frequency (other than 1000 Hz) such that the reconstructed signal would be \( x_d(t) = 0 \). Assume that the reconstruction filter is also changed so that the cutoff frequency of the reconstruction filter is \( \frac{1}{2} \) of the new sampling frequency. (5 points)

2 Possible approaches:

1) Want 600 Hz & 800 Hz to both alias to DC

\[ F_s = \frac{200}{1, 200/1, 200/2, 200/3, 200/4, \ldots} \]

ANSWER: \( F_s = \frac{1400 \text{ or } 200}{\text{ or others}} \)

Is this answer unique? YES \( \text{ or NO} \)

2) Want 800 Hz to alias to 600 Hz

This occurs when \( F_s = 1400 \text{ Hz} \)

(1400/2, 1400/3, 1400/4, etc. also work)

c) Repeat part (a) if the cosines in the original problem are replaced by sines and the sampling frequency is \( F_s = 700 \text{ Hz} \) instead of 1000 Hz. Show all work. (10 points)

\[ x(t) = 8 \sin(2\pi \cdot 100t) \]
5. Sampling, reconstruction, and aliasing (20 points)

Consider the ideal sampling/reconstruction system below:

The input to this system above consists of a desired signal \( x_{\text{desired}}(t) \) and unwanted noise \( x_{\text{noise}}(t) \), i.e. \( x_{\text{in}}(t) = x_{\text{desired}}(t) + x_{\text{noise}}(t) \). You want the desired signal, but not the noise, to appear at the output. The spectra of the signal and noise are both shown below:

![Spectra of the Signal and Noise](image)

a) A computer engineering student suggested that selecting \( F_s = 1000 \) Hz will get rid of the noise, because the noise would fall outside the range of the ideal reconstruction filter. Determine whether this claim is true by finding and sketching the spectrum of the reconstructed signal, \( X_r(j\omega) \). Explain why or why not this system successfully removes the noise as claimed (8 points)

![Spectrum of the Reconstructed Signal](image)

This system is not successful because \( F_s \) doesn't obey Nyquist. \( F_s = 1000 \) Hz and \( F_{\text{max}} = 900 \) Hz so need \( F_s = 1800 \) Hz to obey Nyquist. Thus, for \( X_r(j\omega) \) we get the noise aliased on top of our desired signal.
b) Now assume that \( F_s \) is changed to 2000 Hz. Draw the corresponding \( X_s(j\omega) \) on the figure below and use that figure to determine how the reconstruction filter can be modified such that the reconstructed signal consists only of the desired input signal. (8 points)

\[
X_s(j\omega)
\]

ANSWER: The reconstruction filter should be modified by changing the cutoff frequency to \( 300 \text{Hz} < F_c < 700 \text{ Hz} \).

c) If you had to use the sampling frequency and reconstruction filter given in part (a) (\( F_s = 1000 \text{ Hz} \)), what could you add to the sampling/reconstruction system to assure that the reconstructed signal consists only of the desired input signal? Explain. (4 points)

You could add an anti-aliasing filter (a low pass filter with cutoff frequency \( 300 \text{Hz} < F_c < 700 \text{ Hz} \)). You could also use a bandpass filter with some requirement for upper cutoff and a lower cutoff of \( 0 \text{Hz} < F_{\text{lower}} < 100 \text{ Hz} \).

This anti-aliasing filter allows you to remove the noise before it is sampled, thus preventing aliasing and assuring that the reconstructed signal is only the desired input signal.