EE 351

Sample Exam Solutions

From SP98
1. Complex numbers (10 points)

a) Express the complex number \( z = -4 + j7 \) in polar form. Show all work. (4 points)

\[
|z| = \sqrt{(-4)^2 + 7^2} = 8.06226
\]

\[
\angle z = \tan^{-1} \frac{-7}{-4} = 2.0899 \text{ rad (119.7°)}
\]

Note: If you take \( \tan^{-1} \frac{-7}{-4} \) you get \(-1.05165 \text{ rad}\)
Thus is 4th quadrant \(\Rightarrow\) need to add \(\pi\) to
get 2nd quadrant

b) Determine all cube roots of the complex number \( z = 8e^{j\pi/2} \) and express your answers in rectangular form. Show all work. (6 points)

\[
Z_1 = 8^{1/3} e^{j \frac{\pi/4}{3}} = 2e^{j \frac{\pi}{12}} = 1.9318 + j0.5176
\]

\[
Z_2 = 8^{1/3} e^{j \frac{\pi/4 + 2\pi}{3}} = 2e^{j \frac{9\pi}{12}} = -1.4142 + j1.4142
\]

\[
Z_3 = 8^{1/3} e^{j \frac{\pi/4 + 4\pi}{3}} = 2e^{j \frac{17\pi}{12}} = -0.5176 - j1.9318
\]
2. Signal Transformations (20 points)

Consider the discrete-time sequence given by \( \{x[n]\} = [-4 \ 2 \ 4 \ -2 \ 6] \)

a) Determine the sequence \( \{x[4 - 2n]\} \). (6 points)

\[ x[4 - 2n] = x[-2n + 4] \]

\[
\begin{align*}
\text{Shift } L & \text{ by } 4, \text{ followed by scaling by } 2, \text{ followed by reversal} \\
\text{Shift } L: & \begin{bmatrix} -4 & 2 & 4 & -2 & 6 & 0 & 0 & 0 \end{bmatrix} \\
\text{Scaling by } 2: & \begin{bmatrix} 2 & -2 & 0 & 0 \end{bmatrix} \\
\text{Reversal: } & \begin{bmatrix} 0 & 0 & -2 & 2 \end{bmatrix}
\end{align*}
\]

b) Determine and sketch the even part of \( \{x[n]\} \). (8 points)

\[
EV\{x[n]\} = \frac{x[n] + x[-n]}{2}
\]

\[
\frac{-4 \ 2 \ 4 \ -2 \ 6}{-4 \ 2 \ 10 \ -4 \ 10 \ 2 \ -4}
\]

\[
EV\{x[n]\} = [-2 \ 1 \ 5 \ -2 \ 5 \ 1 \ -2]
\]

\[
\begin{array}{c}
-2 \\
-1 \\
\hline
5 \\
1 \rightarrow n
\end{array}
\]

\[
\begin{array}{c}
-2 \\
-1 \\
\hline
5 \\
1 \rightarrow n
\end{array}
\]

A graph showing the even part of \( x[n] \)

A graph showing the even part of \( x[n] \)

c) Determine the discrete-time sequence \( x[n]\cdot \{ \delta[n+1] + u[n+2] - u[n-1] \} \). (6 points)

\[
\begin{align*}
\text{Break up: } x[n] \delta[n+1] &= x[-1] \delta[n+1] = 4 \delta[n+1] = \begin{bmatrix} 4 & 0 \end{bmatrix} \\
u[n+2] - u[n-1] &= \begin{bmatrix} 2 & 0 \end{bmatrix} \\
\text{So } x[n] \{u[n+2] - u[n-1]\} &= \begin{bmatrix} 2 & 4 & -2 \end{bmatrix} \\
\text{Combining both we get } &= \begin{bmatrix} 2 & 8 & -2 \end{bmatrix}
\end{align*}
\]
3. Aliasing (20 points)

Consider a continuous-time signal \( x_c(t) = 4\cos(2\pi \cdot 800t) + 3\cos(2\pi \cdot 1200t) \) passed through the ideal sampling/reconstruction system below:

\[
\begin{array}{c}
\text{Ideal Sampler} \\
\text{Sampling freq.} = F_s \\
\end{array} \quad \xrightarrow{\text{Ideal Reconstruction Filter}} \quad x_r(t)
\]

a) Determine the output signal \( x_r(t) \) if the sampling frequency is \( F_s = 600 \text{ Hz} \) and the reconstruction filter is an ideal lowpass filter with cutoff frequency = \( F_s/2 \). (5 points)

\[
x_r(t) = 3 + 4\cos(2\pi \cdot 200t)
\]

i.e. 1200 aliased to DC
800 aliases to 200 Hz

b) Determine the sampling frequency \( F_s \) if the reconstructed signal is \( x_r(t) = 7\cos(2\pi \cdot 800t) \) and the reconstruction filter is an ideal lowpass filter with cutoff frequency = \( F_s/2 \). (5 points)

Now i) the 800 Hz component is not aliased
and ii) the 1200 Hz component aliases to 800 Hz

From i) the sampling freq > 1600 Hz
From ii) the sampling freq < 2400 Hz and the difference between the sampling freq and 1200
must be 800 Hz \( \Rightarrow \) \( F_s = 2000 \text{ Hz} \)

Verify:
c) Assuming that an ideal lowpass reconstruction filter is used with cutoff frequency $F_s/2$, determine all possible sampling frequencies $F_s$ such that the reconstructed signal will be a DC signal. (5 points)

Both 800 Hz and 1200 Hz must be integer multiples of $F_s$ for them both to alias to DC. The greatest common factor of (800, 1200) = 400

So $F_s = \frac{400}{n}$  \hspace{1cm} n = 1, 2, 3, ...$

\[ F_s = \frac{400}{n} \quad n = 1, 2, 3, ... \]

---

d) Suppose that the sampling frequency is changed to $F_s = 1500$ Hz, and that an ideal bandpass filter is used for reconstruction. Determine the permissible range of frequencies for the lower edge and the upper edge of this reconstruction filter so that $x_c(t)$ will be reconstructed exactly. If no ranges of frequencies will work, briefly state why. (5 points)

Lower edge 700 Hz < $f_1$ < 800 Hz
Upper edge 1200 Hz < $f_2$ < 1800 Hz

BP filter

Lower edge must fit here
Upper edge must fit here
4. Discrete-time sinusoids (30 points)

a) Consider the discrete-time signal $x_d[n]$ obtained by sampling the continuous-time signal $x(t) = \cos(2\pi \cdot 1000t + \pi / 4)$ at a rate of 3 kHz.

i) Find the discrete-time frequency of $x_d[n]$. (4 points)

\[
x_d[n] = x_c(t) \bigg|_{t=nT} = \cos\left[2\pi \cdot \frac{1000}{3000} n + \frac{\pi}{4}\right] = \cos\left[2\pi \cdot \frac{n}{3} + \frac{\pi}{4}\right]
\]

\[
\Omega = \frac{2\pi}{3} \text{ rad/sample}
\]

ii) Determine if $x_d[n]$ is periodic, and if so, find its fundamental period. (4 points)

\[
\frac{\frac{\Omega}{2\pi}}{\text{ratio of integers}} = \frac{1}{3} \rightarrow \text{period } N = 3
\]

b) Now consider a more general situation. Suppose that we are sampling a continuous-time sinusoid $x(t) = \cos(\omega_0 t)$. Find a condition on the sampling period, $T$, to ensure that the resulting discrete-time sinusoid is periodic. Note: Your condition should depend on $\omega_0$. (8 points)

\[
x_d[n] = \cos(\omega_0 T n) \Rightarrow 2\pi = \omega_0 T
\]

so \[
\frac{\omega_0 T}{2\pi} \text{ must be rational}
\]
c) Again consider sampling \( x(t) = \cos(2\pi \cdot 1000t + \pi / 4) \). Suppose that there are a finite number of choices for sampling period: \( T_k = \frac{1}{3000k} \), \( k = 1, 2, 3, \ldots, 10 \), with the resulting discrete-time signals denoted by \( x_k[n] \), \( k = 1, 2, 3, \ldots, 10 \).

i) Are all of the signals \( x_k \) periodic? Justify your answer. (4 points)

\[
X_k[n] = \cos \left[ \frac{2\pi n}{3k} + \pi / 4 \right] \Rightarrow \Omega_k = \frac{2\pi}{3k}
\]

\[
\frac{\Omega_k}{2\pi} = \frac{1}{3k} \rightarrow always \ rational \ N = 3k
\]

ii) As \( k \) increases, what happens to the rate of oscillation of the discrete-time sinusoid? (3 points)

As \( k \uparrow \), \( \Omega_k \downarrow \), so oscillation rate decreases

iii) As \( k \) increases, what happens to the fundamental period of the discrete-time sinusoid? (3 points)

As \( k \uparrow \), \( N = 3k \uparrow \), so period increases

d) More generally, suppose that \( T_1 > T_2 \), with \( T_1 \) and \( T_2 \) both satisfying the condition in (b), but otherwise arbitrarily chosen. Can you say with certainty that the fundamental period of \( x_1[n] \) is smaller than that of \( x_2[n] \)? Explain. (4 points)

Both \( T_1 \) and \( T_2 \) must satisfy \( \frac{\omega_0 T}{2\pi} = \frac{m_1}{n} \) (ratio of integers)

\[
T_1 = \frac{2\pi m_1}{\omega_0 n_1} \Rightarrow \Omega_1 = \frac{2\pi m_1}{n_1}
\]

\[
T_2 = \frac{2\pi m_2}{\omega_0 n_2} \Rightarrow \Omega_2 = \frac{2\pi m_2}{n_2}
\]

If \( T_1 > T_2 \) then \( \frac{m_1}{n_1} > \frac{m_2}{n_2} \). However, this does not necessarily mean that \( n_1 < n_2 \).

For example, let \( T_1 = \frac{2\pi \cdot 5}{\omega_0 \cdot 3} \), \( T_2 = \frac{2\pi \cdot 3}{\omega_0 \cdot 2} \).

Now \( T_1 > T_2 \) but \( n_1 = 3 \) and \( n_2 = 2 \).
5. Sampling and Reconstruction (20 points)

Consider a sound wave represented by the signal:

\[ x(t) = \cos(2\pi \cdot 5000t) + \cos(2\pi \cdot 15000t) + \cos(2\pi \cdot 30000t) + \cos(2\pi \cdot 45000t) \]

Assume that the audible frequency range of humans is only up to 20 kHz.

a) Consider the sampling and reconstruction system shown below. Suppose that the sampling rate is 40 kHz. Sketch the spectrum of the sampled signal (at point A) and find the reconstructed signal \( x_r(t) \). Assume that the cutoff frequency of the reconstruction filter is \( F_s/2 \). (7 points)

\[ X_s(j\omega) \]

\[ x_r(t) = 2\cos(2\pi \cdot 5000t) + \cos(2\pi \cdot 10000t) + \cos(2\pi \cdot 15000t) \]
b) Now consider a modification of the system in part a). First, an anti-aliasing filter is included in the system before the sampling operation. Second, suppose that we can no longer use ideal filters — both the anti-aliasing and the reconstruction filters are assumed to have the non-ideal frequency characteristic shown below. For this modified system, determine which frequency components will exist in \( x_c(t) \) when \( x(t) \) is the input. (6 points)

Frequency response for both the anti-aliasing and reconstruction filters

After anti-filter: \( 5, 15, 30 \text{ kHz components} \)

After sampling: \( 30 \text{ kHz aliased to 10khz} \)

After reconstruction: \( 5, 10, 15, 30 \) are present

Also: \( 25, 35 \)

c) In parts a) and b), we looked at the reconstructed signal, but this may not be the signal which is actually heard. Consider a model for the frequency characteristic of the human ear, shown below. The signal which is actually heard is the output when the signal \( x_c(t) \) is passed through the ear’s filter. For the given non-ideal anti-aliasing, reconstruction and ear filters, find the minimum sampling rate, \( F_s \), such that the signal which is actually heard will consist only of the audible part of the input signal \( x(t) \). (7 points)

Frequency response of typical human ear

Need aliasing freq > 20 kHz
So 30 kHz component can’t alias to anything less than 20 kHz

\[ F_s - 30 > 20 \Rightarrow F_s > 50 \text{ kHz} \]
5) a) \[ X(t) = \cos(2\pi \cdot 5000t) + \cos(2\pi \cdot 15000t) + \cos(2\pi \cdot 30000t) + \cos(2\pi \cdot 45000t) \]

\[ x(t) = \frac{X(\omega)}{2\pi} \]

\[ X_A(\omega) \]

\[ w \] (kHz)

\[ \omega \]

b) The antialiasing filter removes the 45 kHz component prior to sampling, but some of the 30 kHz component remains, and will cause aliasing after sampling.

So, looking at the sampled spectrum from part a), the components in \( X_r(t) \) will be all the components in the interval \((0, 40)\) kHz, except those due to the 45 kHz signal.

The 45 kHz signal aliases to 5 kHz in \((0, 40)\) (but there is already a 5 kHz term, anyway). Therefore, the components in \( X_r(t) \) occur at 5, 10, 15, 25, 30, and 35 kHz.
Another way to see this easily:

- $5$ sampled $\rightarrow 5, 35$ ($f - \frac{1}{2}F_s$)
- $15$ sampled $\rightarrow 15, 25$
- $30$ sampled $\rightarrow 30, 10$

c) Here, we need to choose the sampling frequency so that all the components which survive the anti-aliasing filter ($5, 15, 30$) have aliases that are all outside the $\langle 0, 20 \rangle$ kHz interval. This way, we will only hear the audible part of the signal -- the $5\mathrm{kHz}$ + $15\mathrm{kHz}$ components.

The basic spectrum lies between $\langle -30, 30 \rangle$ kHz, with the nearest copies lying between $\langle -30 - F_s, 30 - F_s \rangle$ and $\langle -30 + F_s, 30 + F_s \rangle$, to the left and right, respectively. We need to choose $F_s$ so that

$30 - F_s < -20 + -30 + F_s < 20$

or $F_s > 50$

Can also be seen from the picture: