EE 351

MIDTERM #2

Sample exam from FA98.

Note: Some of the notation, terminology, etc. might be different than what we use, since the book was a different one and I try to use the notation in the book each semester.

NOTE:
1. Exam is comprised of 5 problems, each with multiple parts.
2. Point values are given in parentheses for each part.
3. You are allowed one 8 1/2 X 11 sheet of paper and a calculator
4. You must show all work and write your answers in the spaces provided.
5. You have 2 hours to complete the exam.

DO NOT WRITE BELOW THIS LINE:

Problem 1 (16 points) : ____
Problem 2 (34 points) : ____
Problem 3 (20 points) : ____
Problem 4 (15 points) : ____
Problem 5 (15 points) : ____

TOTAL (100 points) : ____
1. YOU MUST EXPLAIN YOUR REASONING FOR EACH PART OF THIS PROBLEM TO RECEIVE CREDIT.

For each discrete-time system below, determine whether it is

i) linear or non-linear
ii) time-invariant or time-varying
iii) causal or non-causal
iv) BIBO stable or unstable

a) \( y[n] = -2y[n-1] + 2n \cdot x[n+1] \) with no initial conditions  (8 points)

i) linear -- difference eqn with no i.c. and no \( x^2, x^3 \) terms, are linear

ii) time varying -- \( x[n+1] \) is time varying

iii) non-causal -- \( y[0] \) depends on \( x[1] \) (future input)

iv) unstable -- \( 2n \cdot x[n+1] \) is unbounded even if \( x[n] \) is bounded

\[
\begin{align*}
\text{b) } y[n] &= \begin{cases} 
x[n] & \text{if } x[n] \geq 1 \\
0 & \text{if } -1 < x[n] < 1 \\
-x[n] & \text{if } x[n] \leq -1 
\end{cases}
\end{align*}
\]

(8 points)

i) non-linear -- obvious from graph. Due to deadband and absolute value

ii) time invariant -- a term shows up only in \( x[n] \)

iii) causal -- \( y[n] \) depends only on \( x[n] \)

iv) stable -- If \( |x[n]|_{\text{max}} = Mx \), \( |y[n]|_{\text{max}} = Mx \) as well
2. You are the manager of the Butterball Turkey Farm in Giblet, PA and you want to predict the turkey farm population using a difference equation. Suppose that during a given month, 25% of the turkeys from the previous month die of natural causes. However, new turkeys are being born each month as well. The birth rate of new turkeys born each month happens to equal 25% of the population from two months ago (since the incubation period for turkey eggs is 2 months). Besides the natural birth/death of turkeys, the turkey farm population is also dependent on the number of turkeys sold to area stores each month, of course.

If we let $t[n]$ represent the number of turkeys at the farm at the end of the nth month, and let $s[n]$ represent the number of turkeys sold to the local stores during month n, the difference equation model for the turkey farm described above is given by $t[n] = 0.75 \cdot t[n - 1] + 0.25 \cdot t[n - 2] - s[n].$

ALL PARTS BELOW ARE INDEPENDENT OF EACH OTHER

a) Draw a Direct Form II system diagram for this population model. (6 points)

(Note – we did not cover this topic in SP03)

b) Define September 1998 as month $n=0$. If the turkey population at the end of July is given by $t[-2] = 5600$, and the population at the end of August is given by $t[-1] = 5000$, calculate by hand (i.e. via iteration) the turkey farm population at the end of each remaining month in 1998.

Assume that the number of turkeys sold from September to December is given by $s[n] = [50 \quad 175 \quad 750 \quad 500]$. (8 points)

$\uparrow$

<table>
<thead>
<tr>
<th>Month</th>
<th>$t_0 = 5600$</th>
<th>$t_1 = 5000$</th>
<th>$t_2 = 4900$</th>
<th>$t_3 = 4200$</th>
<th>$t_4 = 3875$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sep</td>
<td>.75(5600) + .25(5600) - 50 = 5100</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Oct</td>
<td>.75(5100) + .25(5000) - 175 = 4900</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Nov</td>
<td>.75(4900) + .25(5100) - 750 = 4200</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dec</td>
<td>.75(4200) + .25(4900) - 500 = 3875</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
c) Suppose that the number of turkeys sold each month is given by the equation
\[ s[n] = 100 \cdot (0.9)^n \cdot u[n] \]. For this “input signal” determine a closed-form equation for the turkey farm population at the end of month \( n \). IN SOLVING THIS PROBLEM, YOU DO NOT HAVE TO SOLVE FOR ANY OF THE COEFFICIENTS. ONLY THE FORM OF THE EQUATION IS NECESSARY (e.g. \( y[n] = c_1(2)^n u[n] + c_2(-5)^n u[n] + c_3 u[n] \)). (7 points)

Char eqn: \[ E^2 - 0.75 E - 0.25 = 0 \rightarrow (E-1)(E+0.25) = 0 \]
\[ Y_{x=0}[n] \text{ is of form } C_1 u[n] + C_2 (-0.25)^n u[n] \]
\[ Y_{x=0}[n] \text{ is of form } C_1 u[n] - C_2 (-0.25)^n u[n] + A_1 (0.9)^n u[n] \]
\[ y(n) \text{ is of form } C_1 u[n] + C_2 (-0.25)^n u[n] + A_1 (0.9)^n u[n] \]

\[ \text{d) If at the end of August 1998 you suddenly became an animal rights activist and decided to not sell any turkeys (i.e. } s[n] = 0), \text{ use the general solution technique to determine a closed-form equation for the turkey farm population at the end of month } n. \text{ Use the initial conditions from part (b). THIS TIME YOU DO HAVE TO SOLVE FOR THE COEFFICIENTS. Use this equation for } y[n] \text{ to determine the eventual steady-state turkey population on the farm. (9 points)} \]

\[ y[n] = y_{x=0}[n] = C_1 + C_2 (-0.25)^n \]
\[ n = -2: \quad y[-2] = C_1 + C_2 (-0.25)^{-2} = 5600 \rightarrow C_1 + 16C_2 = 5600 \]
\[ n = -1: \quad y[-1] = C_1 + C_2 (-0.25)^{-1} = 5000 \rightarrow C_1 - 4C_2 = 5000 \]
\[ 20C_2 = 600 \quad \text{or} \quad C_2 = 30 \]
\[ C_1 = 5120 \]

\[ y_{x=0}[\infty] = [5120 + 30(-0.25)^n] u[n] \]
\[ y_{x=0}[\infty] = 5120 \]

e) Determine a new difference equation model for this turkey farm if the turkey egg incubation period is 3 months instead of 2 months and only 10% of the turkey population dies each month. Once again, let \( t[n] \) represent the turkey population at the end of the \( n \)th month and let \( s[n] \) represent the number of turkeys sold each month. (4 points)

\[ y[n] = 0.9y[n-1] + 0.25y[n-3] - 5[n] \]
\[ \uparrow \quad \uparrow \quad \uparrow \]
\[ \text{surviving turkeys} \quad \text{new births} \quad \text{sold turkeys} \]
3. YOU MUST SHOW ALL WORK ON THIS PROBLEM TO RECEIVE ANY CREDIT.

Consider an LTI non-recursive system given by its impulse response:

\[ h[n] = a \cdot \delta[n+1] + b \cdot \delta[n] + c \cdot \delta[n-1] + d \cdot \delta[n-3] + e \cdot \delta[n-5] \]

where \( a, b, c, d \) and \( e \) are real constants.

a) Write down the corresponding difference equation for this system. (4 points)

\[ y[n] = a x[n+1] + b x[n] + c x[n-1] + d x[n-3] + e x[n-5] \]

b) What are the restrictions on constants \( a, b, c, d \) and \( e \) so that this system is both causal and stable? (4 points)

Always stable \( \Rightarrow \) FIR system

Causal iff \( a = 0 \)

No restrictions on \( b, c, d, e \)

c) Assume that the impulse response coefficients for this system are given as follows:
\( a = 0, \ b = 0, \ c = 1, \ d = -2, \) and \( e = 3. \) If the input to this system is given by the equation
\( x[n] = (n+2) \cdot \{u[n+1] - u[n-3]\} , \) use graphical (or sliding bar) convolution to determine the output \( y[3], \) where \( y[n] = x[n] * h[n] \). (8 points)

\[ h[n] = \begin{bmatrix} 0 & 1 & 0 & -2 & 0 & 3 \end{bmatrix} \]

\[ x[n] = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \]

\[ y[3] = \begin{bmatrix} 0 & 1 & 0 & -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix} \]

\[ 4 + 0 + -4 + 0 = 0 \]

\[ \downarrow \]

\[ 4+n \]

\[ \uparrow \]

d) Now assume that the impulse response coefficients for this system are given as follows:
\( a = 1, \ b = 0, \ c = 1, \ d = 1 \) and \( e = 0. \) Determine the locations (n values) of the first and last non-zero output points, if the input to this system is given by \( x[n] = [1 \ 1 \ 1 \ 1 \ 1 \ 1]. \) (4 points)

\[ h[n] \text{ exists from } n = -1 \text{ to } n = 3 \]

\[ x[n] \text{ exists from } n = -2 \text{ to } n = 3 \]

So \( y[n] \text{ exists from } n = -3 \text{ to } n = 6 \)
4. YOU MUST SHOW ALL WORK ON THIS PROBLEM TO RECEIVE ANY CREDIT.

Determine $y[n]$ for the discrete-time LTI system given below by using the analytical convolution method. Express your answer in the simplest form possible. (15 points)

$$x[n] = u[-n] + \left(\frac{1}{3}\right)^n \cdot u[n-2] \quad h[n] = \left(\frac{1}{6}\right)^n \cdot u[n] \quad \rightarrow \quad y[n]$$

HINT: Use the distributive property of convolution.

$$y[n] = x[n] * h[n] = u[-n] * \left(\frac{1}{6}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n-2] * \left(\frac{1}{6}\right)^n u[n]$$

Do each convolution separately

$$y_1[n] = \sum_{k=-\infty}^{\infty} u[-k] \left(\frac{1}{6}\right)^{n-k} u[n-k]$$

$$= \sum_{k=-\infty}^{\min(0,n)} \left(\frac{1}{6}\right)^{n-k} = \left(\frac{1}{6}\right)^n \sum_{k=-\infty}^{\min(0,n)} (6)^k$$

- if $n < 0$, $y_1[n] = \left(\frac{1}{6}\right)^n \cdot \frac{(6)^{-\infty} - 6^{n+1}}{1-6} = -\frac{1}{5}\left(\frac{1}{6}\right)^n (-6^{n+1}) = 6/5$
- if $n \geq 0$, $y_1[n] = \left(\frac{1}{6}\right)^n \cdot \frac{(6)^{-\infty} - 6^0}{1-6} = \frac{6}{5}\left(\frac{1}{6}\right)^n$

$$y_2[n] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^k u[k-2] \left(\frac{1}{6}\right)^{n-k} u[n-k] = \left(\frac{2}{3}\right)^n \sum_{k=2}^{\infty} (2)^k$$

$$= \left(\frac{2}{3}\right)^n \frac{(2)^2 - 2^{n+1}}{1-2} = -\left(\frac{1}{3}\right)^n \left[4 - 2 \cdot (2)^n\right] \quad \text{valid for } n \geq 2$$

$$= [2 \left(\frac{1}{3}\right)^n - 4 \left(\frac{1}{6}\right)^n] u[n-2]$$

So $y[n] = \begin{cases} 
\frac{6}{5}, & n < 0 \\
\frac{6}{5} \left(\frac{1}{6}\right)^n, & n = 0, 1 \\
\frac{2}{5} \left(\frac{1}{3}\right)^n - \frac{14}{5} \left(\frac{1}{6}\right)^n, & n \geq 2 
\end{cases}$
5. YOU MUST SHOW ALL WORK ON THIS PROBLEM TO RECEIVE ANY CREDIT.

a) If the output of a discrete-time LTI system is given by

\[ y[n] = \sum_{k=-\infty}^{0} e^{-2k} \cdot x[n-k] \]

determine the impulse response of this system. (5 points)

\[ y[n] = x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k] = \sum_{k=-\infty}^{0} e^{-2k} \cdot x[n-k] = \sum_{k=-\infty}^{0} u[-k] \cdot x(n-k) \]

\[ \Rightarrow h[n] = e^{-2n} \cdot u(-n) \quad \text{OR} \quad \text{If} \ x(n) = \delta(n) \Rightarrow y(n) = h(n) \Rightarrow \]

b) Consider a discrete-time LTI system shown below:

If \( x[n] = u[n+1], \ h_1[n] = \delta[n-1], \ h_2[n] = \delta[n+2], \ h_3[n] = -\delta[n+1], \) and \( h_4[n] = \left(\frac{1}{2}\right)^n \cdot u[n-2], \) determine the system output \( y[n]. \) (7 points)

\[ A = x[n] \cdot h_1[n] = u[n+1] \cdot \delta[n-1] = u[n-1+1] = u[n] \]

\[ B = A \cdot h_2[n] = u[n] \cdot \delta[n+2] = u[n+2] \]

\[ C = A \cdot h_3[n] = u[n] \cdot (-\delta[n+1]) = -u[n+1] \]

\[ D = B + C = u[n+2] - u[n+1] = \delta[n+2] \]

\[ y[n] = D \cdot h_4[n] = \delta[n+2] \cdot \left(\frac{1}{2}\right)^n \cdot u[n-2] \Rightarrow y[n] = \left(\frac{1}{2}\right)^{n+2} \cdot u[n] \]

c) Draw a different combination of \( h_1[n], \) \( h_2[n], \) \( h_3[n], \) and \( h_4[n] \) that would yield the same output as part (b). Note: You do not need the answer in (b) to solve (c). (3 points)

By using the distributive, associative and commutative property, there are several answers, some of them would be:

\[ x[n] \rightarrow h_1[n] \rightarrow h_2[n] \rightarrow y[n] \quad \text{OR} \quad x[n] \rightarrow h_4[n] \rightarrow h_2[n] \rightarrow y[n] \]